Math 301 — Three core theorems for differentiable functions

Summary. In first semester calculus, we learn a technique for finding extrema of functions. We look for critical points of the function—when the function is differentiable everywhere, these are points c where $f'(c) = 0$. Then we check whether we have a maximum or minimum at these points comparing function values at critical points and endpoints. At the root of this technique is the Interior Extremum Theorem. Perhaps just as importantly, this theorem is key in understanding two other fundamental theorems in calculus, Rolle's Theorem and the Mean Value Theorem. The following problems walk you through proving these three theorems.

Theorem 1 (Interior Extremum Theorem)**.** *Let f be differentiable on the open interval (a, b). If f achieves a maximum value at some point $c \in (a, b)$, then $f'(c) = 0$. The same holds if $f(c)$ is a minimum value.*

Problem 1. In the following questions, which outline a sequential proof of the Interior Extremum Theorem, assume that f is differentiable on the open interval (a, b) and assume that f achieves a maximum value at the point $c \in (a, b)$. That is, assume that $f(c) \geq f(x)$ for all $x \in (a, b)$. The case where $f(c)$ is a minimum value is similar and left for you to think about on your own.

a. It's straightforward to construct a sequence (x_n) so that $a < x_n < c$ for all $n \geq 1$ and $\lim x_n = c$. For example, by the Archimedean property, there exists N_0 such that $0 < 1/N_0 < c - a$ and so $a < c - 1/N_0$. Then we could define $x_n = c - 1/(n + N_0)$ for all $n \geq 1$. Notice $x_n \in (a, c)$ for all $n \geq 1$ and $\lim x_n = c$. Try to also make a sequence (y_n) so that $c < y_n < b$ for all $n \geq 1$ and $\lim y_n = c$.

b. Is the difference quotient
$$\frac{f(x_n) - f(c)}{x_n - c}$$
positive or negative for each $n \geq 1$? Try thinking about the sign of the numerator and denominator separately.

c. Is the difference quotient
$$\frac{f(y_n) - f(c)}{y_n - c}$$

positive or negative for each $n \geq 1$?

d. What do each of the previous parts tell you about the sign of $f'(c)$? Write a concluding statement to explain why the result follows.

Theorem 2 (Rolle's Theorem). *Let f be a function that is continuous on $[a, b]$ and differentiable on (a, b). If $f(a) = f(b)$ then there exists a point $c \in (a, b)$ where $f'(c) = 0$.*

Problem 2. In this problem, which outlines how to prove Rolle's theorem, we suppose that f is continuous on $[a, b]$, differentiable on (a, b), and $f(a) = f(b)$. Recall that the Extreme Value Theorem tells us that there exist x_0 and y_0 in $[a, b]$ so that

$$f(y_0) \leq f(x) \leq f(x_0)$$

for all $x \in [a, b]$. That is, there are points x_0 and y_0 where the maximum and minimum (ie. the extreme points) of the function are achieved. These extreme points might occur at the endpoints or the interior points of the interval $[a, b]$.

a. Suppose both x_0 and y_0 occur at the endpoints of $[a, b]$. What can you say about the function? Why does there exist $c \in (a, b)$ such that $f'(c) = 0$.

b. Suppose at least one of x_0 or y_0 occurs in (a, b). Why does there exist $c \in (a, b)$ such that $f'(c) = 0$.

Theorem 3 (Mean Value Theorem). *Let f be a function that is continuous on $[a, b]$ and differentiable on (a, b). Then there exists a point $c \in (a, b)$ so that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Problem 3. We now attempt to prove the Mean Value Theorem.

a. Let $L(x)$ be the secant line that connects the points $(a, f(a))$ and $(b, f(b))$ on the graph of f. What is the slope of $L(x)$? That is, what is $L'(x)$?

b. Let $g(x) = f(x) - L(x)$. What is the value of $g(a)$? What about $g(b)$? Explain why g satisfies the hypotheses of Rolle's Theorem.

c. By Rolle's Theorem, there exists $c \in (a, b)$ such that $g'(c) = 0$. Why does this imply the desired conclusion of the Mean Value Theorem?