Math 301 — Fundamental theorem of calculus

Summary. In first semester calculus, we learn a technique called the Fundamental Theorem of Calculus for computing definite integrals using antiderivatives. We compute an antiderivative of the function of interest, find its values at the endpoints of the region of integration, and subtract the results. The questions below try to get you to prove this theorem and see that it's really a consequence of uniform continuity and the mean value theorem.

Theorem 1 (Fundamental Theorem of Calculus, part II)**.** *Let f be continuous on the closed interval $[a, b]$. Define a new function $g : [a, b] \to \mathbb{R}$, called the* ***area function****, by the formula*

$$g(x) = \int_a^x f(t)\, dt.$$

Then g is differentiable on (a, b) and $g'(x) = f(x)$.

Problem 1. In the following questions, which walk us through writing a proof of the theorem above, assume that f is continuous on $[a, b]$ and x_0 is an arbitrary element of (a, b). Our ultimate goal is to give an ϵ-δ proof that shows

$$\lim_{x \to x_0} \frac{g(x) - g(x_0)}{x - x_0} = f(x_0).$$

As we open our proof, we should let $\epsilon > 0$ and tell the reader that we aim to show there exists $\delta > 0$ such that when $0 < |x - x_0| < \delta$, we have

$$\left| \frac{g(x) - g(x_0)}{x - x_0} - f(x_0) \right| < \epsilon. \tag{1}$$

a. Answer the following questions as a group.

1. Before working with the left hand side of (1), we should explain where our candidate for δ comes from. Why is f uniformly continuous? Complete the following: *The uniform continuity of f gives us that there exists $\delta > 0$ so that when $0 < |x - x_0| < \delta$ we have...*

2. We're now ready to do the main computation of our proof, which I'll show below. Make sure you know why each step works, but write explanations for the labeled lines. *Observe that when* $0 < |x - x_0| < \delta$,

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\begin{aligned}
\left| \frac{g(x) - g(x_0)}{x - x_0} - f(x_0) \right| &= \left| \frac{\int_a^x f(t)\,dt - \int_a^{x_0} f(t)\,dt}{x - x_0} - f(x_0) \right| \\
&= \left| \frac{\int_{x_0}^x f(t)\,dt}{x - x_0} - \frac{f(x_0)(x - x_0)}{x - x_0} \right| \\
&= \left| \frac{\int_{x_0}^x f(t)\,dt}{x - x_0} - \frac{\int_{x_0}^x f(x_0)\,dt}{x - x_0} \right| \\
&= \left| \frac{1}{x - x_0} \int_{x_0}^x (f(t) - f(x_0))\,dt \right| \\
&\leq \frac{1}{|x - x_0|} \int_{x_0}^x |f(t) - f(x_0)|\,dt \qquad (2) \\
&< \frac{1}{|x - x_0|} \int_{x_0}^x \epsilon\,dt \qquad (3) \\
&= \epsilon.
\end{aligned}
$$

b. For your homework submission, put together a complete proof of the theorem that incorporates your answers to the questions above.

Theorem 2 (Fundamental Theorem of Calculus, part I)**.** *Let f be a function that is continuous on $[a, b]$ and let F be any antiderivative of f. This means F is a function that is differentiable on (a, b) such that $F'(x) = f(x)$ for all $x \in (a, b)$. Then*

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Problem 2. In this problem, which outlines how to prove the theorem above, we suppose that f is continuous on $[a, b]$ and F is an arbitrary antiderivative of f on (a, b).

a. Answer the following questions as a group:

1. Explain why g is an antiderivative of f.
2. Explain the relationship between g and F using the Mean Value Theorem.
3. Use part 2 to explain why $F(b) - F(a) = \int_a^b f(x)\,dx$.

b. For your homework submission, use your answers to the previous questions to write a proof of the theorem.