Math 301 — Uniform convergence and differentiation

Problem 1. Consider the sequence of functions defined by

$$g_n(x) = \frac{x^n}{n}.$$

a. Find $g(x) = \lim_{n\to\infty} g_n(x)$ for all $x \in [0,1]$. Show that $g_n \to g$ uniformly on $[0,1]$.

b. Find $g'(x)$ for all $x \in [0,1]$.

c. Show that $(g_n'(x))$ converges for each $x \in [0,1]$ and let $h(x) = \lim_{n\to\infty} g_n(x)$ for each $x \in [0,1]$. Are h and g' the same? Does $g_n' \to h$ uniformly on $[0,1]$?

Problem 2. Consider the sequence of functions

$$h_n(x) = \sqrt{x^2 + \frac{1}{n}}.$$

a. Compute the pointwise limit h of (h_n) and then prove that $h_n \to h$ uniformly on $\mathbb{R}$.

b. Take as given that h_n is differentiable on $\mathbb{R}$ for each $n \in \mathbb{N}$. Show that $g(x) = \lim_{n\to\infty} h_n'(x)$ exists for all $x \in \mathbb{R}$. Explain how we know that for every $\epsilon > 0$ the sequence (h_n') does not converge uniformly to g on $(-\epsilon, \epsilon)$.

Problem 3. Consider the sequence of functions

$$f_n(x) = \frac{x}{1 + nx^2}.$$

a. It turns out that for each $n \in \mathbb{N}$ the function f_n achieves a maximum and a minimum value somewhere on $\mathbb{R}$. Use calculus to find the points on $\mathbb{R}$ where each f_n achieves its maximum and minimum values. Use this to prove that $f_n \to f$ uniformly on $\mathbb{R}$ for some function f. What is f?

b. Compute $f_n'(x)$ and find all the values of x for which $f'(x) = \lim_{n\to\infty} f_n'(x)$.