Math 301 — Series of functions

Problem 1. Let $0 < a < 1$. Prove that $\sum_{n=0}^{\infty} x^n$ converges uniformly on $[-a, a]$. What function does this series converge to?

Problem 2. Prove that $\sum_{n=0}^{\infty} x^n$ does not converge uniformly on $(-1, 1)$. *Hint: recall that if (f_n) is a sequence of functions each bounded on A and $f_n \to f$ uniformly on A then f is bounded on A.*

Problem 3. Decide whether each of the following statements is true or false, providing a short justification or counterexample as appropriate.

a. If $\sum_{n=1}^{\infty} g_n$ converges uniformly on A, then (g_n) converges uniformly to zero on A.

b. If $0 \leq f_n(x) \leq g_n(x)$ for all $x \in A$ and $\sum_{n=1}^{\infty} g_n$ converges uniformly on A, then $\sum_{n=1}^{\infty} f_n$ converges uniformly on A.

c. If $\sum_{n=1}^{\infty} f_n$ converges uniformly on A, then there exists a sequence (M_n) such that $|f_n(x)| \leq M_n$ for all $x \in A$ and the series $\sum_{n=1}^{\infty} M_n$ converges.

Problem 4. Let $f(x) = \sum_{n=1}^{\infty} n^{-2} \sin(n^2 x)$. This is sometimes referred to as Riemann's function and turns out to be differentiable *almost nowhere*. Taking as given that $\sin x$ is continuous on $\mathbb{R}$ prove that f is continuous on $\mathbb{R}$.

Problem 5. Let $f(x) = \sum_{n=1}^{\infty} 2^{-n} \cos(2^n x)$. Taking as given that $\cos x$ is continuous on $\mathbb{R}$ prove that f is continuous on $\mathbb{R}$.

Remark 1. The function in the previous problem is known as the *Weierstrass' Monster Function.* It is a function which is continuous but *nowhere* differentiable on $\mathbb{R}$. Take a look at the *Weierstrass function Wikipedia article* to see a depiction of the graph of f. Take a look at the *Quanta Magazine article The Jagged, Monstrous Function That Broke Calculus* for some history of its creation. Before Karl Weierstrass published his 1872 paper on it, arguably most mathematicians did not believe such functions existed or at least dismissed them as useless (or worse). Henri Poincaré called it *an outrage against common sense.* Charles Hermite called it *a deplorable evil.* Émile Picard said that if Isaac Newton had known about it, he would have never created his theory of calculus. Since then, such functions proved essential in Albert Einstein's theory of Brownian motion, Kyoshi Ito's theory of stochastic integrals, and Helge von Koch's work on fractal geometry, and more.