Math 301 —Uniform convergence and integrals

Summary. What happens when we start to think about integrals of sequences of functions. Is it true that

$$\lim_{n\to\infty}\int_a^b f_n(x)\,dx = \int_a^b \lim_{n\to\infty} f_n(x)\,dx?$$

There are two limits that are being considered in such a question: the limit in terms of n and the limit in terms of the partition size of the Riemann sum of integral. The question is really asking, can we switch the order in which we do these limits? The answer in general is no, but uniform convergence will save the day!

Problem 1. Let $f_n(x) = nx^n$.

a. Find $f(x) = \lim_{n\to\infty} f_n(x)$ for $x \in [0, 1)$.

b. Does $f_n \to f$ uniformly on $[0, 1)$? Can you prove your claim?

c. Compute $\int_0^1 f_n(x)\,dx$ and compute $\int_0^1 f(x)\,dx$.

d. For this example, state whether the following equation is true:

$$\lim_{n\to\infty}\int_0^1 f_n(x)\,dx = \int_0^1 \lim_{n\to\infty} f_n(x)\,dx.$$

Problem 2. Let's remind ourselves of some inequalities involving integrals. For each of the following determine whether $\square$ should be replaced with $\leq$ or $\geq$.

a. Let $c > 0$ and suppose that $f : [a, b] \to \mathbb{R}$ is a continuous function such that $0 \leq f(x) \leq c$ for all $x \in [a, b]$. Then

$$\int_a^b f(x)\,dx \;\square\; \int_a^b c\,dx.$$

b. Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then

$$\left|\int_a^b f(x)\,dx\right| \square \int_a^b |f(x)|\,dx.$$

Thinking about the integral as a limit of Riemann sums, why do you think this relation is called the triangle inequality for integrals?

Problem 3. The theorem below states that the order of limits and integrals can be interchanged when we have continuous functions that converge **uniformly**. Finish the proof by pasting together the following expressions in the correct order and using $\leq$, $<$, or $=$ in between each. When you're finished, talk about why we required continuity on a closed, as opposed to open, interval.

a. $\int_a^b |f_n(x) - f(x)| \, dx$

b. $\left| \int_a^b f_n(x) \, dx - \int_a^b f(x) \, dx \right|$

c. ϵ

d. $\int_a^b \frac{\epsilon}{b-a} \, dx$

e. $\left| \int_a^b (f_n(x) - f(x)) \, dx \right|$

Theorem. *Let (f_n) be a sequence of continuous functions and suppose that $f_n \to f$ uniformly on the closed interval $[a, b]$. Then*

$$\lim_{n \to \infty} \int_a^b f_n(x) \, dx = \int_a^b f(x) \, dx.$$

Proof. Let $\epsilon > 0$. Because $f_n \to f$ uniformly, there exists N such that

$$|f_n(x) - f(x)| < \frac{\epsilon}{b - a}$$

for all $n > N$ and $x \in [a, b]$. Observe that for all $n > N$... $\square$

Problem 1

a) $f(x) = \lim_{n\to\infty} nx^n = 0$ when $x \in [0,1)$.

b) $f_n \not\to f$ uniformly on $[0,1)$

Proof Consider

$$g_n(x) = |f_n(x) - f(x)| = nx^n$$

and $g_n'(x) = n^2 x^{n-1}$. Notice $g_n'(x) > 0$ for all x, n so g_n is an increasing function and its supremum is achieved as we approach 1. That is,

$$\sup\{|f_n(x) - f(x)| : x \in [0,1)\} = g_n(1) = n$$

Therefore

$$\lim_{n\to\infty} \sup\{|f_n(x) - f(x)| : x \in [0,1)\} = \lim_{n\to\infty} n$$
$$= \infty \neq 0.$$

c) $$\int_0^1 f_n(x)\,dx = \int_0^1 n x^n\,dx = \frac{n}{n+1}$$

$$\int_0^1 f(x)\,dx = \int_0^1 0\,dx = 0$$

d) It's not true since

$$\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \lim_{n\to\infty} \frac{n}{n+1} = 1$$

and $$\int_a^b \lim_{n\to\infty} f_n(x)\,dx = \int_a^b 0\,dx = 0.$$

Problem 2 Both inequalities should be $\leq$.

The second is called the triangle inequality since it uses our old triangle inequality

$$|x_1 + \cdots + x_n| \leq |x_1| + \cdots + |x_n|.$$

The reason is as follows.

Notice $$\int_a^b f(x)\,dx = \lim_{n\to\infty} \underbrace{\sum_{i=1}^{n} f(x_i)\,\Delta x}_{\text{Riemann sum}}$$

Therefore $$\left|\int_a^b f(x)\,dx\right|$$

$$= \lim_{n\to\infty} \left|\sum_{i=1}^{n} f(x_i)\,\Delta x\right|$$

$$\overset{\text{triangle ineq.}}{\leq} \lim_{n\to\infty} \sum_{i=1}^{n} \left|f(x_i)\,\Delta x_i\right|$$

$$= \int_a^b |f(x)|\,dx.$$

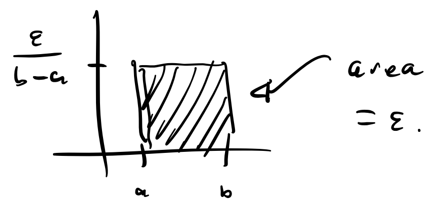
Problem 3

$$\left| \int_a^b f_n(x)\,dx - \int_a^b f(x) \right|$$

$$= \left| \int_a^b \left(f_n(x) - f(x) \right) dx \right|$$

$$\leq \int_a^b \left| f_n(x) - f(x) \right| dx$$

$$< \int_a^b \frac{\varepsilon}{b-a}$$



$$= \varepsilon$$