Math 301 — Uniform convergence and derivatives

Summary. What happens when we start to think about derivatives of sequences of functions. Is it true that

$$\lim_{n\to\infty} f_n' = \left(\lim_{n\to\infty} f_n\right)'?$$

There are two limits that are being considered in such a question: the limit in terms of n and the limit in terms of the the difference quotient that defines the derivative. The question is really asking, can we switch the order in which we do these limits? The answer in general is no, but uniform convergence will again make things work!

In Problem 2 below it will be helpful to recall the following theorem.

Theorem (Fundamental Theorem of Calculus)**.** *Let $f : [a, b] \to \mathbb{R}$ be a continuous function.*

a. If F is any function such that $F' = f$, then $\int_a^b f(x)\,dx = F(b) - F(a)$.

b. Define $g(x) = \int_a^x f(t)\,dt$. Then g is differentiable on (a, b) and $g'(x) = f(x)$.

Problem 1. Can you give an example of a sequence (f_n) of differentiable functions where

$$\lim_{n\to\infty} f_n' \neq \left(\lim_{n\to\infty} f_n\right)'?$$

Before answering this, can you think of a sequence of differentiable functions whose pointwise limit is not differentiable?

Theorem. *Let (f_n) be a sequence of differentiable functions with domain (a, b) and suppose that f_n' is continuous for each n, f_n' converges uniformly, and there exists $x_0 \in (a, b)$ such that $\lim f_n(x_0)$ exists. Then*

a. $\lim f_n(x)$ *exists for all* $x \in (a, b)$,

b. $\lim f_n(x)$ *is a differentiable function,*

c. $\lim f_n' = (\lim f_n)'$.

Problem 2. The following questions walk you through proving the theorem above. The idea of the proof is to exploit the Fundamental Theorem of Calculus to turn this into a question about interchanging limits and integrals, for which we've already built a theorem in the last worksheet. Assume that $x \in (a, b)$ and $n \geq 1$ are arbitrary.

a. Explain why
$$\int_{x_0}^{x} f_n'(t)\, dt = f_n(x) - f_n(x_0).$$

b. Let $h = \lim_{n\to\infty} f_n'$. Explain why
$$\lim_{n\to\infty} (f_n(x) - f_n(x_0)) = \int_{x_0}^{x} h(t)\, dt.$$

c. Explain why $\lim_{n\to\infty} f_n(x)$ exists and let $f(x) = \lim_{n\to\infty} f_n(x)$. Show that
$$f(x) = f(x_0) + \int_{x_0}^{x} h(t)\, dt.$$

d. Explain why f is differentiable and $f'(x) = h(x)$. Do you see that we've now proved all three parts of the theorem?