Math 301 — Exam 3 review guide

Your third exam will be a self-scheduled exam during the exam period May 9-13. The in-class portion will contain 6 problems. One problem will ask you to provide definitions to some terms. One problem will ask you to provide an example for a given description. One problem will be a true/false question. There will be three proof questions. The exam will cover material from Homework 8 to Homework 10. In the textbook, this is material spanning Sections 4.6-6.4, excluding sections not covered in lecture. I have outlined some important definitions, theorems, and general topics below. Also, the problems below give you a sampling of some problems like those that will appear on the exam, but it's not necessarily comprehensive, so make sure to review old homework, quizzes, worksheets, and lecture notes.

Definitions and theorems

While not necessarily comprehensive, here is an absolutely-must-know list of definitions.

- continuous function, uniformly continuous function, differentiable function
- pointwise convergence, uniform convergence

Similarly, here is a list of statements you should know.

- sequential criterion for continuity, algebraic continuity theorem, discontinuity criterion
- preservation of compact sets by continuous functions, extreme value theorem, sequential criterion for absence of uniform continuity
- differentiable functions are continuous, algebraic differentiability theorem, Mean Value Theorem
- continuous limit theorem, differentiable limit theorem

Proofs and logic

You should know the following about proofs and logic. You should know how to:

- prove a given function is continuous (or not continuous) or uniformly continuous (or not uniformly continuous) or differentiable (or not differentiable) on a given set
- apply the Mean Value Theorem to prove basic facts about differentiable functions
- prove a given sequence of functions converges pointwise or converges uniformly on a given set

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Sample problems

These problems are not comprehensive and there are more than you will see on the exam itself, but will give you an idea of the kinds of questions to expect.

Problem 1. Please state whether the following statements are true or false. If a statement is true, explain why. If a statement is false, give a counterexample and modify the statement slightly to make it true.

a. Let $f : \mathbb{R} \to \mathbb{R}$ be a given function. If f^3 is continuous at 0, then f is continuous at 0.

b. Let $f : \mathbb{R} \to \mathbb{R}$ be a given function. If $f(K)$ is compact for every compact set $K \subseteq \mathbb{R}$ then f is continuous on $\mathbb{R}$.

c. If f is continuous at 0 then f is differentiable at 0.

d. If f and g are differentiable at 0 and $g(0) \neq 0$, then f/g is differentiable at 0.

e. Let (f_n) be a given sequence of functions defined on a set A. If for every $\epsilon > 0$ and every $x \in A$ there exists $N \in \mathbb{N}$ such that $|f_n(x) - f_m(x)| < \epsilon$ for all $n, m \geq N$ then (f_n) converges uniformly on A.

Problem 2. Please do the following proof problems.

a. Let f be continuous on $\mathbb{R}$ and let $V = \{x \in \mathbb{R} : f(x) = 0\}$. Prove that V is closed. Recall that a set A is closed if for every $(x_n) \subseteq A$ such that $x_n \to a$ we have that $a \in A$.

b. Take as given that $f(x) = \sqrt{x}$ is uniformly continuous on $[1, \infty)$. Prove that f is uniformly continuous on $[0, \infty)$.

c. Let $f : \mathbb{R} \to \mathbb{R}$ be a given function. Suppose that $f(0) = 0$ and there exists $\alpha > 1$ and $\beta > 0$ such that $|f(x)| \leq |x|^\alpha$ for all $x \in (-\beta, \beta)$. Prove that f is differentiable at 0.

d. Let f be twice differentiable on an interval A which contains 0. Let $(x_n) \subseteq A$ be a sequence with $x_n \to 0$ such that $x_n \neq 0$ for all $n \in \mathbb{N}$. Prove that if $f(x_n) = 0$ for all $n \in \mathbb{N}$ then $f(0) = 0$, $f'(0) = 0$, and $f''(0) = 0$.

e. Let $g_n(x) = (nx + x^2)/(2n)$. Prove that (g_n) converges uniformly on $[-M, M]$ for any $M > 0$ but does not converge uniformly on $\mathbb{R}$.