Math 301 — Consequences of completeness

Problem 1. Let $S = \{1 - 1/n : n \in \mathbb{N}\}$. Use the Archimedean property to prove that $\sup S = 1$. To get you started, let me suggest two possible approaches, listed below. Please feel free to use either.

a. Prove that 1 is an upper bound of S and then prove that $1 \leq u$ for any upper bound u of S.

b. Use the lemma from last time which states that $\sup S = \alpha$ if and only if for every $\epsilon > 0$ there exists $x \in S$ such that $x > \alpha - \epsilon$.

Theorem (Nested Interval Property)**.** *For each $n \in \mathbb{N}$, assume we are given a closed interval $I_n = [a_n, b_n] = \{x \in \mathbb{R} : a_n \leq x \leq b_n\}$. Assume also that each I_n contains I_{n+1}. That is, assume that the intervals are nested as follows:*

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots.$$

Then $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$.

Problem 2. This problem outlines a proof of the Nested Interval Property, which is stated above.

a. Let $A = \{a_n : n \in \mathbb{N}\}$ be the collection of left endpoints of the closed intervals I_n.

   1. Explain why A is bounded above. Do this by listing out all upper bounds of A that you can identify.
   2. Explain why A has a supremum.

b. Let $x = \sup A$.

   1. Explain why $a_n \leq x$ for all $n \in \mathbb{N}$.
   2. Explain why $x \leq b_n$ for all $n \in \mathbb{N}$.

c. Explain why $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$.

Problem 3. For each $n \in \mathbb{N}$, let $J_n = (0, 1/n) = \{x \in \mathbb{R} : 0 < x < 1/n\}$.

a. Are these intervals nested? If so, in which order?

b. Make a conjecture about their intersection $\bigcap_{n=1}^{\infty} J_n$ and prove your conjecture using the Archimedean property.

c. Why are your answers to these questions not in conflict with the Nested Interval Property?