

## Math 301 — $\mathbb{Q}$ is dense in $\mathbb{R}$

*Summary.* We've learned the Archimedean property and its consequences. Now we'd like to use this idea to prove that between any two real numbers, we must be able to find a rational number. This fact is commonly referred to as the fact that the rationals are dense in the reals.

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**Theorem** ( $\mathbb{Q}$  is dense in  $\mathbb{R}$ ). *For every  $a, b \in \mathbb{R}$  such that  $a < b$ , there exists a rational number  $m/n \in \mathbb{Q}$  such that  $a < \frac{m}{n} < b$ .*

*Note 1.* Through the rest of the worksheet, we'll assume  $a, b > 0$  for simplicity.

*Note 2.* The idea of our proof will be to show that there is a “mesh” of rational numbers that is fine enough so that there is at least one rational number between  $a$  and  $b$ . That is, (1) we want a number  $n$  so that a multiple of  $1/n$  can land between  $a$  and  $b$  and (2) we want to show that such a multiple  $m$  can be found.

**Problem 1.** Explain why there exists  $n \in \mathbb{N}$  such that  $1/n$  is smaller than the distance between  $a$  and  $b$ .

**Problem 2.** Suppose we have  $n \in \mathbb{N}$  such that  $1/n$  is smaller than the distance between  $a$  and  $b$ . Explain why there exists  $m \in \mathbb{N}$  such that  $m > na$ .

**Problem 3.** Suppose we have  $n \in \mathbb{N}$  such that  $1/n$  is smaller than the distance between  $a$  and  $b$ , and suppose that we also have the *smallest*  $m \in \mathbb{N}$  such that  $m > na$ . This means  $m - 1 < na$ . Why can we conclude that  $a < \frac{m}{n} < b$ ?

**Problem 4.** Explanations to the previous problems are the core of the proof of the theorem. Let's put everything together in a formal proof. Fill in the blanks below to give a complete proof.

*Proof.* Our aim is to find  $m, n \in \mathbb{N}$  so that  $a < m/n < b$ . To do this we must first establish that there exists  $n \in \mathbb{N}$  so that the increments of size  $1/n$  are not too big. By BLANK1, there exists  $n \in \mathbb{N}$  such that

$$\frac{1}{n} < \text{BLANK2}. \quad (1)$$

Next, we'll show that we can find a multiple of  $1/n$  that fits between  $a$  and  $b$ . By BLANK3 there exists  $m \in \mathbb{N}$  such that

$$m > na. \quad (2)$$

We choose  $m$  to be the smallest such integers so that

$$m - 1 < an. \quad (3)$$

Observe that

$$\begin{aligned} a &< \frac{m}{n} \\ &< \frac{an + 1}{n} \\ &= a + \frac{1}{n} \\ &< b, \end{aligned}$$

where the first inequality is due BLANK4, the second is due to BLANK5, and the third is due to BLANK6. Thus, the inequalities above show that  $a < \frac{m}{n} < b$ .  $\square$