Math 301 — Monotone convergence theorem

Summary. The monotone convergence theorem is useful as a tool for proving a sequence converges, even when we don't necessarily know the value it converges to. In the first example below, we will see it's even useful when we can guess the limit of the sequence but it's difficult to find a closed form expression for the terms of the sequence.

Problem 1. Consider the sequence given by $a_1 = 9$ and $a_{n+1} = \frac{1}{2}\left(a_n + \frac{9}{a_n}\right)$ for all $n \geq 1$.

a. Suppose the sequence converges to a value L. Find L.

b. Make a conjecture for whether the sequence is increasing or decreasing.

c. If you believe the sequence is increasing, make a conjecture for an upper bound for the sequence. If you believe it is decreasing, make a conjecture for a lower bound for the sequence.

d. Prove your conjectured bound using induction.

e. Prove your increasing/decreasing conjecture.

f. Make a conclusion.

Problem 2. The monotone convergence theorem can be used to prove series converge, which is useful since we often don't know the value of infinite sums. Here, we outline a proof that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges. Let $m \geq 1$ and consider the *partial sum sequence*

$$s_m = \sum_{n=1}^{m} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \cdots + \frac{1}{m^2}.$$

Recall that a series converges if its partial sum sequence converges. That is, if $\lim s_m$ exists, then $\sum \frac{1}{n^2}$ converges.

a. Explain why (s_m) is increasing.

b. It's straightforward to see that $n^2 > n(n-1)$ and, using a little algebra, it's possible to show that

$$\frac{1}{n(n-1)} = \frac{1}{n-1} - \frac{1}{n}$$

for all $n \geq 1$. Use these facts to show that (s_m) is bounded above by 2. If you get stuck and need a reminder about the idea of a telescoping series, ask me.

c. Make a conclusion.