Math 301 — Monotone Convergence Theorem and Infinite Series

Problem 1. The following questions get us thinking about convergence and divergence of infinite series.

a. True or false: if $b_n > 0$ for all $n \in \mathbb{N}$, then the sequence of partial sums of the series $\sum_{n=1}^{\infty} b_n$ is monotone.

b. True or false: if $b_n > 0$ for all $n \in \mathbb{N}$, then the sequence of partial sums of the series $\sum_{n=1}^{\infty} b_n$ is bounded.

c. Let $m \in \mathbb{N}$ such that $m \geq 2$. Consider the telescoping sum

$$\sum_{n=2}^{m} \frac{1}{n(n-1)} = \sum_{n=2}^{m} \left(\frac{1}{n-1} - \frac{1}{n} \right).$$

Simplify the sum above as much as possible in terms of m.

d. Which is the correct inequality for all $n \geq 2$:

$$\frac{1}{n^2} < \frac{1}{n(n-1)} \quad \text{or} \quad \frac{1}{n^2} \geq \frac{1}{n(n-1)}?$$

e. Let (s_m) be the partial sum sequence of the series $\sum_{n=1}^{\infty} 1/n^2$. Use the Monotone Convergence Theorem to prove that (s_m) converges and thus that $\sum_{n=1}^{\infty} 1/n^2$ converges.

f. What can we conclude about the value $\sum_{n=1}^{\infty} 1/n^2$ converges to?