Math 339SP, Spring 2022 — Exam 1

Mount Holyoke College

Due March 11 at 5:00 pm

Instructions. This exam consists of 4 questions, with multiple parts, for a total of 50 points. Please do all of them. To receive full credit, you must show your work and provide details and justification where appropriate. You may use your class notes, the textbook, any materials posted to the class web page, and R. However, you may not use other resources (eg. other textbooks or sites on the internet other than our class web page), and I ask that you avoid discussing any aspect of the exam with anyone (except me, Tim). Please submit your work on Gradescope.

Note. I know that you've all been working hard, both in this class and outside of it. I want you to be proud of the effort that you've put in, proud of your individual growth so far, and proud of your integrity. Part of that means taking the honor code seriously, and working on this exam by yourself, without any collaboration or help from classmates or outside materials. I write all of this not because I suspect that you'll cheat, but because I want you to know that I value you each as individuals and value your work and ideas, right or wrong.

Problem 1 (15 points)**.** A bike share program in a certain town has hubs at 7 locations where residents can check out a bike and return it to any hub once finished with their ride. The transition matrix for where bikes are typically returned after one ride, given their check-out location, is given below.

$$P = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \\ f \\ g \end{array} \begin{array}{c} \begin{array}{ccccccc} a & b & c & d & e & f & g \end{array} \\ \left[\begin{array}{ccccccc} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1/4 & 0 & 0 & 3/4 & 0 & 0 & 0 \\ 0 & 1/4 & 0 & 3/4 & 0 & 0 & 0 \\ 0 & 0 & 1/4 & 0 & 3/4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/4 & 3/4 \\ 0 & 0 & 0 & 0 & 0 & 2/5 & 3/5 \end{array}\right] \end{array}.$$

1. Find $\lim_{n\to\infty} P^n$ without technology and justify each entry.
2. Explain the long term behavior of the system in its real world context. That is, interpret your results in the previous part. What kind of city infrastructure might cause this behavior?
3. Suppose that we follow a bike that starts at hub f. Find the expected number of check-outs until the bike returns to hub f in two distinct ways: (1) using your previous work in this problem and (2) using first-step analysis.

Problem 2 (10 points)**.** For each of the following, either give an example of a Markov chain with the desired properties or explain why it's not possible to give an example. When an example can be given, give a short justification for its validity. You may exhibit the chain in any way that makes the example clear (eg. giving the transition matrix, giving the transition state diagram with edges labeled, declaring it to be a random walk on a given graph).

1. A Markov chain which has a limiting matrix but not a limiting distribution.
2. A Markov chain which does not have a limiting matrix.
3. A Markov chain which is not irreducible but has a limiting distribution.

Problem 3 (10 points)**.** A 2-state Markov chain $X_0, X_1, X_2, \dots$ has transition matrix given by

$$P = \begin{matrix} & a & b \\ a & 1-p & p \\ b & q & 1-q \end{matrix},$$

where $p, q > 0$. We define a new chain $Z_n = (X_n, X_{n-1})$, for $n \geq 1$, which keeps track of states on consecutive steps.

1. Give the state space for the new chain.
2. Prove that the new chain has a limiting distribution by computing limits (ie. without using the Limit Theorem for Regular Matrices or the Fundamental Limit Theorem for Ergodic Markov Chains).

Problem 4 (15 points)**.** Let $X_0, X_1, X_2, \dots$ be a Markov chain with finite state space $\mathcal{S}$ and transition matrix P. Among the following 9 statements, identify the 3 pairs of equivalent statements. Give a short justification for the equivalence of each pair. Note that statements A and B are equivalent if $A \Rightarrow B$ and $B \Rightarrow A$ are both true implications.

1. $P_{ii} > 0$ for all $i \in \mathcal{S}$.
2. $P_{ij} > 0$ for all $i, j \in \mathcal{S}$.
3. $P_{ij}^n > 0$ for all $i, j \in \mathcal{S}$ and all $n \geq 1$.
4. For each $i, j \in \mathcal{S}$ there exists $n \geq 1$ such that $P_{ij}^n > 0$.
5. There exists $n \geq 1$ such that for all $i, j \in \mathcal{S}$, $P_{ij}^n > 0$.
6. There exists a probability distribution π such that $\pi P = \pi$.
7. There exists an initial distribution α so that the distribution of X_n is α for all $n \geq 1$.
8. $X_0, X_1, X_2, \dots$ is an irreducible Markov chain.
9. $X_0, X_1, X_2, \dots$ is an aperiodic Markov chain.