Math 339SP, Fall 2025 — Homework 4

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Due October 2 at 5:00 pm

Instructions. This problem set contains problems mostly from Week 4 of class. The problem numbers refer to our textbook, *Introduction to Stochastic Processes with R* by Robert P. Dobrow.

Problem 1. Please do the following textbook problems: 3.13, 3.17, 3.28, 3.29, 3.34, 3.35.

Problem 2. Consider the Markov chain with state space $\mathcal{S} = \{1, 2, 3, 4, 5, 6\}$ and transition matrix

$$P = \begin{bmatrix} 0.5 & 0.5 & 0 & 0 & 0 & 0 \\ 0.3 & 0.7 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.1 & 0 & 0.9 & 0 \\ 0.25 & 0.25 & 0 & 0 & 0.25 & 0.25 \\ 0 & 0 & 0.7 & 0 & 0.3 & 0 \\ 0 & 0.2 & 0 & 0.2 & 0.2 & 0.4 \end{bmatrix}.$$

a. Give the communication classes of the Markov chain and classify each state as recurrent or transient. No need to justify your answers.

b. Rewrite P in canonical form.

c. Without using technology, find the long term probability that the Markov chain will be in

   1. state 1, given that it starts in state 1
   2. state 6, given that it starts in state 6.

Remark 1. In Exercise 3.13 and Problem 2, you are asked to rewrite P in canonical form. This means you must reorder the rows and columns of P, as well as rearrange the corresponding probabilities in each entry of the matrix, so that the first rows consist of transient states and the subsequent rows have states in the same recurrent communication class listed consecutively. Please read Example 3.14 for a concrete example of this and notice that the canonical form is a block-upper-triangular matrix.

Remark 2. In Exercise 3.17, to find a closed form expression for P^n just means to find a general formula for the entries of P^n in terms of n. Use the recurrence and transience criteria in the box on page 98 to respond to the second part of the question.

Remark 3. In Exercise 3.28, please make sure to justify the values of each entry in $\lim_{n\to\infty} P^n$ using the theory we've built up. As the question notes, you should do this without using technology. You should be able to do this without any computation. Please note that the matrix Q given by

$$Q = \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 2/3 & 0 & 1/3 \\ 0 & 2/3 & 1/3 \end{bmatrix}$$

is regular since $Q^2 > 0$.

Remark 4. In Exercise 3.34, please assume that the chain has finite state space. We will introduce the term *ergodic Markov chain* soon. When a Markov chain has finite state space, the term ergodic simply means that the chain is irreducible (i.e. has one communication class) and aperiodic (i.e. all states have period 1). We will also introduce the following fact, which is used implicitly in the problem statement, soon: any irreducible, finite-state Markov chain has a unique stationary distribution. You may skip the final part of the question since this was already addressed in Exercise 2.19.