Math 339SP, Spring 2024 — Homework 5

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Due March 1 at 5:00 pm

Instructions. This problem set covers material from Week 5 of class, with a focus on Chapter 3 of the textbook.

Problem 1. We will soon learn a theorem called the Limit Theorem for Ergodic Markov chains. We define a Markov chain to be *ergodic* if it has finitely many states, is irreducible (ie. one communication class), and is aperiodic (all states have period 1). The theorem states that an ergodic Markov chain has a limiting distribution (whose components are all positive and which is the unique stationary distribution of the Markov chain). The theorem serves as a better version of the Limit Theorem for Regular Matrices since irreducibility and aperiodicity are hypotheses that can be checked without having to resort to investigating powers of the transition matrix, and thus are generally more practical. That is, it is sometimes impractical to write down the transition matrix for a Markov chain and then prove that it's regular. Explain why the following Markov chains are ergodic and thus have a limiting distribution.

1. The lazy random walk on a connected graph with finitely many vertices.
2. A Markov chain with state space $\{1, \ldots, k\}$ that evolves as follows. Before each step, a coin is tossed. If it lands heads, the system stays in its current state. If it lands tails, it moves to a different state with all $k-1$ other states equally likely.
3. A Markov chain on a graph (where each vertex has at least one neighbor, but the graph has possibly disconnected components) that evolves as follows. Before each step, a coin is tossed. If it lands heads, the next state is chosen according to the usual rules of a random walk: the walker moves to one of its neighbor vertices at random. If the coin lands tails, the next state is chosen uniformly at random with all vertices in the graph equally likely.

Problem 2. Consider a Markov chain with transition matrix P and two recurrent communication classes C_1 and C_2, with 3 states and 4 states respectively. Suppose that the transition probabilities among C_1 are described by the regular 3×3 matrix P_1, and, similarly, transitions among C_2 are described by the regular 4×4 matrix P_2. Let $1/a, 1/b, 1/c$ be the expected first return times for the states in C_1 and let $1/d, 1/e, 1/f, 1/g$ be the expected first return times for the states in C_2. Find as many entries of $\lim_{n\to\infty} P^n$ as possible.

Problem 3. Try the following exercises from Chapter 3.

1. Exercise 3.11
2. Exercise 3.15 (Note that a chess board has 64 squares. Also, note that the knight is allowed to make moves in an 'L' shape. Try thinking of the Markov chain in this problem as a random walk on a graph, where the vertices are the 64 squares and there is an edge connecting two vertices if the knight is allowed to move from one of those squares to the other. It will be important to count the degrees of all the vertices in this graph in order to find the stationary distribution of this random walk. Recall the general formula for the stationary distribution of a random walk on a graph in Example 3.8.)
3. Exercise 3.18
4. Exercise 3.19