§4.3 Probability Generating Functions

Let X be a discrete random variable taking values in $\{0, 1, 2, \dots\}$. The probability generating function of X is the function

$$G(s) = E[s^X]$$

$$= \sum_{k=0}^{\infty} s^k P(X = k)$$

defined for all values of s where the series converges.

Example if $X \sim \text{Unif}\{0, 1, 2\}$, then

$$G(s) = E[s^X]$$

$$= s^0 P(X=0) + s^1 P(X=1) + s^2 P(X=2)$$

$$= \frac{1}{3} + \frac{1}{3}s + \frac{1}{3}s^2$$

Proposition $P(X = k) = \dfrac{G^{(k)}(0)}{k!}$ for all $k \geq 0$.

Proof Note

$$G(s) = P(X=0) + sP(X=1) + s^2 P(X=2) + \dots$$

So $G(0) = P(X=0)$. Moreover

$$G'(s) = P(X=1) + 2sP(X=2) + 3s^2 P(X=3) + \dots$$

which implies $G'(0) = P(X=1)$. Also,

$$G''(s) = 2P(X=2) + 6P(X=3) + 4s^3 P(X=4) + \cdots$$

which implies $G''(0) = 2P(X=2)$. Continuing in this way we get

$$G^{(k)}(0) = k!\, P(X=k).$$

Conclusion The distribution of X is completely determined by its probability generating function.

Proposition Let $n \geq 1$ and $X_1, \dots, X_n$ i.i.d random variables with probability generating function $G(s) = E[s^{X_i}]$. Let $Z = X_1 + \cdots + X_n$ have prob. generating function $G_Z(s)$. Then

$$G_Z(s) = [G(s)]^n.$$

Proof Observe that

$$\begin{aligned} G_Z(s) &= E[s^Z] \\ &= E[s^{X_1 + \cdots + X_n}] \\ &= E\left[\prod_{i=1}^{n} s^{X_i}\right] \\ &= \prod_{i=1}^{n} E[s^{X_i}] \quad \text{(by independence)} \\ &= \prod_{i=1}^{n} G(s) \\ &= G(s)^n. \end{aligned}$$

In the Galton-Watson process context

Let $G_n(s)$ denote the pgf of Z_n for each $n \geq 0$ and let $G(s)$ denote the pgf of the offspring dist.

<u>Proposition</u> For all $n \geq 1$, $G_n(s) = G_{n-1}(G(s))$.

<u>Reminder</u> $E[X \mid Y = y]$ is some function $g(y)$ whose domain is the range of Y and $E[X \mid Y] = g(Y)$ is a random variable. The Law of Total Expection ($E[X] = \sum_y E(X \mid Y=y) P(Y=y)$) can be formulated as $E[X] = E[E[X \mid Y]]$.

<u>Proof of Proposition</u>

Since $Z_n = \sum_{k=1}^{Z_{n-1}} X_k$,

$$G_n(s) = E\left(s^{\sum_{k=1}^{Z_{n-1}} X_k} \right)$$

$$= E\left(E\left(s^{\sum_{k=1}^{Z_{n-1}} X_k} \,\Big|\, Z_{n-1} \right) \right)$$

by the Law of Total Expectation (see Section 1.5 for review)

Moreover

$$E\left(s^{\sum_{k=1}^{Z_{n-1}} X_k} \,\middle|\, Z_{n-1} = z \right)$$

$$= G(s)^z$$

which implies

$$E\left(s^{\sum_{k=1}^{Z_{n-1}} X_k} \,\middle|\, Z_{n-1} \right)$$

$$= G(s)^{Z_{n-1}}$$

and so

$$G_n(s) = E\left[G(s)^{Z_{n-1}} \right]$$

$$= G_{n-1}\left(G(s) \right).$$

Corollary If $Z_0 = 1$, then $G_0(s) = s$,

$$G_1(s) = G_0(G(s)) = G(s)$$

$$G_2(s) = G_1((G(s)) = G(G(s))$$

$$G_3(s) = G_2(G(s)) = G(G(G(s)))$$

and in general $G_n(s) = \underbrace{G \circ \dots \circ G}_{n\text{-fold composition}}(s) = G(G_{n-1}(s))$