§ 7.3 Infinitesimal generator

Recall that $P_{ij}(t) = P(X_t = j \mid X_0 = i)$ is the transition function of a CTMC $(X_t)_{t \geq 0}$.

Our goal today is to understand how to compute $P(t)$.

Theorem (Kolmogorov forward equation) The transition function $P(t)$ satisfies the differential equation

$$P'(t) = P(t) Q$$

where $Q = P'(0)$.

Proof Note that for any $t, h > 0$,

$$\begin{aligned} P_{ij}(t+h) &= P(X_{t+h} = j \mid X_0 = i) \\ &= \sum_{k \in S} P(X_{t+h} = j \mid X_t = k) P(X_t = k \mid X_0 = i) \\ &= \sum_{k \in S} P_{ik}(t) P_{kj}(h) \\ &= [P(t) P(h)]_{ij} \end{aligned}$$

Therefore, $P(t+h) = P(t) P(h)$ (this is called the Chapman-Kolmogorov equation). Next, note that

$$\begin{aligned} P'(t) &= \lim_{h \to 0^+} \frac{P(t+h) - P(t)}{h} \\ &= \lim_{h \to 0^+} \frac{P(t)P(h) - P(t)}{h} \end{aligned}$$

$$= P(t) \lim_{h \to 0^+} \frac{P(h) - I}{h}$$

$$= P(t)\, P'(0)$$

$$= P(t)\, Q.$$

Def Given a matrix A, the matrix e^A given by

$$e^A = I + \sum_{k=1}^{\infty} \frac{1}{k!} A^k$$

is called the *matrix exponential*.

Theorem $P(t) = e^{tQ}$ solves the Kolmogorov Forward Equation.

Proof Observe that if $P(t) = e^{tQ}$, then

$$P'(t) = \frac{d}{dt}\left(e^{tQ}\right)$$

$$= \frac{d}{dt}\left(I + tQ + \frac{t^2}{2!} Q^2 + \frac{t^3}{3!} Q^3 + \cdots \right)$$

$$= Q + tQ^2 + \frac{t^2}{2!} Q^3 + \frac{t^3}{3!} Q^4 + \cdots$$

$$= \left(I + tQ + \frac{t^2}{2!} Q^2 + \frac{t^3}{3!} Q^3 + \cdots \right) Q$$

$$= e^{tQ}\, Q$$

$$= P(t)\, Q.$$

Definition The matrix $Q = P'(0)$ is called the infinitesimal generator of the Markov chain.

Lemma Let Q be the infinitesimal generator and let $i \in S$.

1. for each $j \neq i$, Q_{ij} is the mean transition rate from i to j.
2. $Q_{ii} = -q_i$, where q_i is the hold time parameter of i.

Proof. ① Let $j \in S$ with $j \neq i$. Then

$$
\begin{aligned}
Q_{ij} &= P_{ij}'(0) \\
&= \lim_{h \to 0^+} \frac{P_{ij}(h) - P_{ij}(0)}{h} \\
&= \lim_{h \to 0^+} \frac{P_{ij}(h)}{h} \\
&= \lim_{h \to 0^+} \frac{P(1 \text{ transition to } j \text{ in } [0,h])}{h} \\
&= \lim_{h \to 0^+} \frac{E[\#\text{ of transitions to } j \text{ in } [0,h]]}{h} \\
&= q_{ij}
\end{aligned}
$$

② Observe that

$$
\begin{aligned}
Q_{ii} &= P_{ii}'(0) \\
&= \lim_{h \to 0^+} \frac{P_{ii}(h) - P_{ii}(0)}{h} \\
&= \lim_{h \to 0^+} \frac{P_{ii}(h) - 1}{h} \\
&= \lim_{h \to 0^+} \frac{-(1 - P_{ii}(h))}{h}
\end{aligned}
$$

$$= \lim_{h \to 0^+} \frac{-\sum_{j \neq i} P_{ij}(h)}{h} \quad \text{(since ith row sum is 1)}$$

$$= -\sum_{j \neq i} \lim_{h \to 0^+} \frac{P_{ij}(h)}{h}$$

$$= -\sum_{j \neq i} q_{ij}$$

$$= -q_i .$$

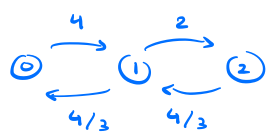
Problem 1. Consider two independent machines that are maintained by a single person. Each machine functions for an exponentially distributed amount of time before breaking down. On average each machine functions for a half hour before breaking down. The repair time for either machine is exponentially distributed. The average repair time is 45 minutes. Assume that at time $t = 0$ (8:00 am) neither machine is broken. Find the following probabilities.

a. Both machines are broken at 10:30 am.

b. Neither machine is broken at 11:00 am.

c. One machine is broken at 2:15 pm.

d. The long term probabilities that 0, 1, or 2 machines are broken.



(a) $P_{02}(2.5)$

(b) $P_{00}(3)$

(c) $P_{01}(6.25)$

$$Q = \begin{array}{c c} & \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \end{array} & \begin{pmatrix} -4 & 4 & 0 \\ 4/3 & -10/3 & 2 \\ 0 & 4/3 & -4/3 \end{pmatrix} \end{array}$$

Problem 1

```{r}
Q = matrix(c(-4,4,0,
             4/3,-10/3,2,
             0,4/3,-4/3), nrow = 3, ncol = 3, byrow = TRUE)
colnames(Q) = 0:2
rownames(Q) = colnames(Q)
expm(2.5*Q)["0", "2"]  # part a
expm(3*Q)["0", "0"]    # part b
expm(6.25*Q)["0", "1"] # part c
expm(10*Q)["0",]       # part d
                       # (note t = 10 is long enough to reach equilibrium)
```

```
[1] 0.5273675
[1] 0.1179263
[1] 0.3529423
        0         1         2
0.1197039 0.3591118 0.5386677
```

Problem 2. Consider a population where each member acts independently and takes an exponentially distributed amount of time, on average 6 months, to produce an offspring. Further, suppose that the lifespan of each member is exponentially distributed, with an average lifespan of 4 years. Finally suppose that when the population size is 5, offspring are no longer produced. Let X_t be the population size at time t in years and suppose that $X_0 = 1$. Find the distribution of the population size and its mean after

a. at 3 months

b. at 1 year

c. at 18 months

$$\begin{tikzcd}
0 \arrow[r, bend left, "0"] & 1 \arrow[l, bend left, "1/4"] \arrow[r, bend left, "2"] & 2 \arrow[l, bend left, "2/4"] \arrow[r, bend left, "4"] & 3 \arrow[l, bend left, "3/4"] \arrow[r, bend left, "6"] & 4 \arrow[l, bend left, "1"] \arrow[r, bend left, "8"] & 5 \arrow[l, bend left, "5/4"]
\end{tikzcd}$$

$$Q = \begin{array}{c|cccccc|}
 & 0 & 1 & 2 & 3 & 4 & 5 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1/4 & -2.25 & 2 & 0 & 0 & 0 \\
2 & 0 & 1/2 & -4.5 & 4 & 0 & 0 \\
3 & 0 & 0 & 3/4 & -6.75 & 6 & 0 \\
4 & 0 & 0 & 0 & 1 & -9 & 8 \\
5 & 0 & 0 & 0 & 0 & 5/4 & -5/4
\end{array}$$

```
## Problem 2

```{r}
Q = matrix(c(0,0,0,0,0,0,
             .25,-2.25,2,0,0,0,
             0,.5,-4.5,4,0,0,
             0,0,.75,-6.75,6,0,
             0,0,0,1,-9,8,
             0,0,0,0,1.25,-1.25), nrow = 6, ncol = 6, byrow = TRUE)
colnames(Q) = 0:5
rownames(Q) = colnames(Q)
expm(0.25*Q)["2",]          # part a distribution
sum(expm(0.25*Q)["2",]*0:5) # part a mean
expm(1*Q)["2",]             # part b distribution
sum(expm(1*Q)["2",]*0:5)    # part b mean
expm(1.5*Q)["2",]           # part c distribution
sum(expm(1.5*Q)["2",]*0:5)  # part c mean
```
```

```
        0         1         2         3         4         5
0.0023216 0.0563669 0.3638740 0.2722903 0.1594020 0.1457453
[1] 2.96732
        0         1         2         3         4         5
0.0111507 0.0294616 0.0453654 0.0646872 0.1460173 0.7033177
[1] 4.414912
        0         1         2         3         4         5
0.0137232 0.0135619 0.0181260 0.0348326 0.1336155 0.7861409
[1] 4.619478
```