§ 2.3 Basic computations

Theorem Let $(X_n)_{n\geq 0}$ be a Markov chain with transition matrix P with state space $S=\{1,\dots,m\}$. Then the matrix P^n contains the n-step transition probabilities for each $n\geq 1$. In other words,

$$(P^n)_{ij} = P(X_n=j \mid X_0=i)$$

for each $i,j\in S$.

Proof Let

$$P = \begin{pmatrix} p_{11} & p_{12} & \cdots & p_{1m} \\ p_{21} & & & \\ \vdots & & & \\ p_{m1} & & \cdots & p_{mm} \end{pmatrix}$$

Then the ij-entry of P^2 is given by

$$(P^2)_{ij} = \begin{pmatrix} p_{i1} & p_{i2} \cdots & p_{im} \end{pmatrix} \begin{pmatrix} p_{1j} \\ p_{2j} \\ \vdots \\ p_{mj} \end{pmatrix}$$

$$= p_{i1}p_{1j} + p_{i2}p_{2j} + \cdots + p_{im}p_{mj}$$

$$= \sum_{k=1}^{m} p_{ik}\,p_{km}.$$

On the other hand,

$$P(X_2=j \mid X_0=i) = \sum_{k=1}^{m} P(X_2=j \mid X_1=k, X_0=i)\,P(X_1=k \mid X_0=i)$$

$$= \sum_{k=1}^{m} p_{kj}\,p_{ik} = (P^2)_{ij}$$

Next,

$$P(X_3=j \mid X_0=i) = \sum_{k=1}^{m} P(X_3=j \mid X_2=k, X_0=i)\, P(X_2=k \mid X_0=i)$$

$$= \sum_{k=1}^{m} P_{kj}\,(P^2)_{ik}$$

$= ij$-entry of the matrix product $P^2 \cdot P = P^3$

The general n-step case can be proved by induction. Try it if you know about induction proofs.

Theorem Let $\alpha = (\alpha_1, \dots, \alpha_m)$ be a probability vector that gives the initial distribution of the Markov chain. That is, $P(X_0 = i) = \alpha_i$ for each $i \in S$. Then αP^n gives the distribution of X_n for each $n \geq 0$.

Proof Observe that αP is an m-element row vector and the jth entry is

$$(\alpha P)_j = (\alpha_1, \alpha_2, \dots, \alpha_m) \begin{pmatrix} P_{1j} \\ P_{2j} \\ \vdots \\ P_{mj} \end{pmatrix}$$

$$= \sum_{k=1}^{m} \alpha_k P_{kj}$$

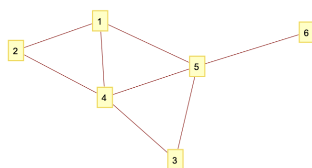
$$= \sum_{k=1}^{m} P(X_0 = k)\, P(X_1 = j \mid X_0 = k)$$

$= P(\bar{X}_1 = j)$

Therefore, αP gives the distribution of $\bar{X}_1$.

Try proving the $n=2$ case for yourself.

Do the general case if you know about induction proofs.



Let's go back to the random walk on a graph from last time using the same graph, which is shown above.

Problem 1. Express the following probabilities in terms powers of the transition matrix P and its entries. Then use R to compute their values. The matrix_powers.Rmd file on the class web page gives a quick introduction to this.

a. $P(X_7 = 5 \mid X_4 = 4)$

b. $P(X_{50} = 5 \mid X_{40} = 2)$

c. $P(X_{12} = 3 \mid X_4 = 1)$

(a) P^3_{45}

(b) P^{10}_{25}

(c) P^8_{13}

```
```{r}
P = matrix(c(0, 1/3, 0, 1/3, 1/3, 0,
             1/2, 0, 0, 1/2, 0, 0,
             0, 0, 0, 1/2, 1/2, 0,
             1/4, 1/4, 1/4, 0, 1/4, 0,
             1/4, 0, 1/4, 1/4, 0, 1/4,
             0, 0, 0, 0, 1, 0), nrow = 6, ncol = 6, byrow = T)
(P %^% 3)[4,5]
(P %^% 10)[2,5]
(P %^% 8)[1,3]
```
```

```
[1] 0.2552083
[1] 0.2575114
[1] 0.1314305
```

Problem 2. Let's go back to the random walk on a graph using the graph shown above. Suppose the random walker starts at a random vertex according to the probability vector $\alpha = (0.1, 0.2, 0.05, 0.35, 0.2, 0.1)$. By this we mean $P(X_0 = i) = \alpha_i$ for each $i = 1, \ldots, 6$. Express the following probabilities in terms of α and P. Then use R to compute their values. The command `alpha = c(0.1,0.2,0.05,0.35,0.2,0.1)` will let you make a row vector.

a. $P(X_7 = 5)$

b. $P(X_{25} = 3)$

c. $P(X_{17} = 3 \mid X_9 = 2)$

d. Challenge questions:

   1. $P(X_{50} = 5, X_{40} = 2)$
   2. $P(X_{12} = 3, X_4 = 1, X_2 = 3)$

(a) $(\alpha P^7)_5$

```{r}
alpha = c(0.1, 0.2, 0.05, 0.35, 0.2, 0.1)
(alpha %*% (P %^% 7))[5]
(alpha %*% (P %^% 25))[3]
(P %^% 8)[2,3]
```

```
[1] 0.2483326
[1] 0.1250011
[1] 0.1208733
```

(b) $(\alpha P^{25})_3$

(c) P^8_{23}

(d) (1) $(\alpha P^{40})_2 \, P^{10}_{25}$

(2) $(\alpha P^2)_3 \, P^2_{31} \, P^8_{13}$