§ 3.2 Stationary Distributions

Goal Find limiting distribution by

① proving it must exist

② solving the equation $xP = x$.

Definition Let P be the transition matrix for a given Markov chain. A probability vector π is called a stationary distribution for the Markov chain if it solves the equation $xP = x$.

Remark Here's the meaning behind the name. If π is a stationary distribution and $X_0 \sim \pi$, then for any $n \geq 1$

$$\begin{aligned} X_n \sim \pi P^n &= (\pi P) P^{n-1} \\ &= \pi P^{n-1} \\ &= (\pi P) P^{n-2} \\ &= \pi P^{n-2} \\ &= \dots = \pi P = \pi. \end{aligned}$$

So the distribution of the chain at each time step remains the same if the initial distribution was a stationary distribution.

Def A transition matrix P is called <u>regular</u> if there exists $n \geq 1$, such that $P^n > 0$ (ie all entries of P^n are <u>strictly</u> positive for some $n \geq 1$).

Theorem (Limit Theorem for Regular Markov Chains)

If a Markov chain has a regular transition matrix then it has a limiting distribution λ <u>and</u> the entries of λ are strictly positive. Moreover, λ is the unique stationary distribution of the Markov chain.

Remark This addresses our goal but later we will discuss an equivalent characterization of regular matrices that is more probabilistic in nature. We'll hold off discussing a proof until then.

Example Find the limiting distribution of the 3-state chain with transition matrix

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{pmatrix} 1/5 & 3/5 & 1/5 \\ 1/10 & 4/5 & 1/10 \\ 1/10 & 3/5 & 3/10 \end{pmatrix} \end{array}$$

Notice P is regular, so it has a limiting distribution and it will satisfy the system

$$\begin{cases} xP = x \\ x_1 + x_2 + x_3 = 1 \end{cases}$$

$$\Longleftrightarrow \begin{cases} \frac{1}{5}x_1 + \frac{1}{10}x_2 + \frac{1}{10}x_3 = x_1 \\ \frac{3}{5}x_1 + \frac{4}{5}x_2 + \frac{3}{5}x_3 = x_2 \\ \frac{1}{5}x_1 + \frac{1}{10}x_2 + \frac{3}{10}x_3 = x_3 \\ x_1 + x_2 + x_3 = 1 \end{cases}$$

$$\Longleftrightarrow \begin{cases} -\frac{4}{5}x_1 + \frac{1}{10}x_2 + \frac{1}{10}x_3 = 0 \\ \frac{3}{5}x_1 - \frac{1}{5}x_2 + \frac{3}{5}x_3 = 0 \\ \frac{1}{5}x_1 + \frac{1}{10}x_2 - \frac{7}{10}x_3 = 0 \\ x_1 + x_2 + x_3 = 1 \end{cases}$$

We can solve this by finding the reduced row echelon form of the augmented matrix

$$\left(\begin{array}{ccc|c} -\frac{4}{5} & \frac{1}{10} & \frac{1}{10} & 0 \\ \frac{3}{5} & -\frac{1}{5} & \frac{3}{5} & 0 \\ \frac{1}{5} & \frac{1}{10} & -\frac{7}{10} & 0 \\ 1 & 1 & 1 & 1 \end{array}\right)$$

$$\xrightarrow{\text{RREF}} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1/9 \\ 0 & 1 & 0 & 3/4 \\ 0 & 0 & 1 & 5/36 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

So the unique stationary distribution (and limiting distribution) is $\pi = \left(\frac{1}{9}, \frac{3}{4}, \frac{5}{36}\right)$.

Problem 2. Consider the Markov chain with transition matrix

$$P = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/3 & 2/3 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Find all stationary distributions of P by solving $\pi P = \pi$. Use R to find the reduced row echelon form of an appropriate augmented matrix. Remember that sometimes a linear system of equations has infinitely many solutions and you need to introduce the notion of free variables to parametrize the solution set.

π_3 is "free variable"

$$\left[\begin{array}{ccc|c} & (P-I)^T & & \begin{matrix}0\\0\\0\end{matrix} \\ 1 & 1 & 1 & 1 \end{array}\right] \xrightarrow[\text{in } R]{\text{RREF}} \left[\begin{array}{ccc|c} 1 & 0 & 2/5 & 2/5 \\ 0 & 1 & 3/5 & 3/5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\Rightarrow \begin{cases} \pi_1 + 2/5\,\pi_3 = 2/5 \\ \pi_2 + 3/5\,\pi_3 = 3/5 \\ \pi_3 = t \end{cases}$$

$\Rightarrow$ Infinite set of stationary distributions of the form

$$\left\{ \left(\frac{2}{5} - \frac{2}{5}t,\ \frac{3}{5} - \frac{3}{5}t,\ t \right) : 0 \le t \le 1 \right\}$$

Problem 3. Repeat the previous problem using

$$P = \begin{bmatrix} 1/2 & 1/2 & 0 & 0 \\ 1/3 & 2/3 & 0 & 0 \\ 0 & 0 & 4/5 & 1/5 \\ 0 & 0 & 1/4 & 3/4 \end{bmatrix}.$$

π_4 is free variable

$$\left[\begin{array}{cccc|c} & & (P-I)^T & & \begin{matrix}0\\0\\0\\0\end{matrix} \\ 1 & 1 & 1 & 1 & 1 \end{array}\right] \xrightarrow[\text{in } R]{\text{RREF}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 9/10 & 2/5 \\ 0 & 1 & 0 & 27/20 & 3/5 \\ 0 & 0 & 1 & -5/4 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\Rightarrow \begin{cases} \pi_1 + \frac{9}{10}\pi_4 = 2/5 \\ \pi_2 + \frac{27}{20}\pi_4 = 3/5 \\ \pi_3 = 5/4\,\pi_4 \\ \pi_4 = t \end{cases} \Rightarrow$$

Infinite set of stationary distributions of the form

to ensure prob. vector

$$\left\{ \left(\frac{2}{5} - \frac{9}{10}t,\ \frac{3}{5} - \frac{27}{20}t,\ \frac{5}{4}t,\ t \right) : 0 \le t \le \frac{4}{9} \right\}$$