§3.3, 3.5 Class properties and periodicity

Theorem (recurrence and transience are class properties)
The states of a communication class are either all recurrent or all transient.

Proof Let i and j be states in the same communication class. Suppose i is recurrent. We must show j is recurrent. That is, we must show $\sum_{n=0}^{\infty} P_{jj}^n = \infty$. Since i and j communicate, there exist $r, s \geq 0$ such that $P_{ij}^r > 0$ and $P_{ji}^s > 0$. Let $n \geq 0$. Then

$$\begin{aligned} P_{jj}^{r+s+n} &= P(\overline{X}_{r+s+n} = j \mid \overline{X}_0 = j) \\ &\geq P(\overline{X}_s = i, \overline{X}_{s+n} = i, \overline{X}_{r+s+n} = j \mid \overline{X}_0 = j) \\ &= P_{ji}^s P_{ii}^n P_{ij}^r. \end{aligned}$$

Therefore,

$$\begin{aligned} \sum_{n=0}^{\infty} P_{jj}^{r+s+n} &\geq \sum_{n=0}^{\infty} P_{ji}^s P_{ii}^n P_{ij}^r \\ &= P_{ji}^s \left(\sum_{n=0}^{\infty} P_{ii}^n \right) P_{ij}^r = \infty \end{aligned}$$

which implies $\sum_{n=0}^{\infty} P_{jj}^n = \infty$ since $\sum_{n=0}^{\infty} P_{jj}^n = \sum_{n=0}^{r+s-1} P_{jj}^n + \sum_{n=0}^{\infty} P_{jj}^{r+s+n}$.

Now suppose i is transient. Then j must be transient too. Otherwise, if j were recurrent, then we'd be able to prove i was recurrent too by repeating the previous argument

Theorem If a Markov chain is irreducible and has a finite state space, then all states are recurrent.

Proof Let $S = \{1, 2, \ldots, m\}$ denote the Markov chain's state space. Since the chain is irreducible, it has one communication class and so either all states are recurrent or all states are transient. Suppose they're all transient. Then if the chain starts in state 1, it must eventually not return and will live among states $2, 3, \ldots, m$. But then the chain will eventually never return to state 2 and live only among $3, \ldots, m$. Eventually, since the state space is finite, there will be no states left that can be visited.

Theorem A recurrent communication class is **closed** (meaning if C is a communication class with recurrent states and $i \in C$, $j \notin C$, then $P_{ij} = 0$).

Canonical Form of P Suppose our Markov chain has recurrent comm. class $R_1, \ldots, R_m$. Since these classes are closed, they give rise to stochastic matrices $P_1, \ldots, P_m$ which describe transitions and states in $R_1, \ldots, R_m$ respectively. Reordering the states as necessary, P has the block form $P = \begin{array}{c} \\ T \\ R_1 \\ \vdots \\ R_m \end{array} \begin{array}{c} \begin{array}{cccc} T & R_1 & \cdots & R_m \end{array} \\ \begin{pmatrix} * & * & & \\ 0 & P_1 & & 0 \\ \vdots & & \ddots & \\ 0 & & & P_m \end{pmatrix} \end{array}$ where T denotes the set of transient states.

Def The period of a state i, denoted $d(i)$, is given by $d(i) = \gcd\{n > 0 : P_{ii}^n > 0\}$. In words, it's the greatest divisor of the set of possible return times to i. If $d(i) = 1$, state i is called aperiodic. If the set of possible return times of i is empty, we declare $d(i) = +\infty$.

Examples See worksheet.

Theorem The states of a comm. class all have the same period. In other words, periodicity is a class property.

Proof Let $i \sim j$ be in the same communication class. We'll try to show $d(i) = d(j)$ by showing $d(i) \leq d(j)$ and $d(j) \leq d(i)$. We'll show that $d(i) \leq d(j)$ by showing that $d(i)$ is a divisor of the set $\{n > 0 : P_{jj}^n > 0\}$ and than exploit the fact that $d(j)$ is the greatest divisor of this set. To show that $d(i)$ is a divisor of this set, we must that for any $n > 0$ such that $P_{jj}^n > 0$, there exists $m \in \mathbb{Z}$ such that $d(i)m = n$.

Let $n > 0$ such that $P_{jj}^n > 0$. Since $i \sim j$, there

exist $r, s \geq 0$ such that $P_{ij}^r > 0$ and $P_{ji}^s > 0$.

We claim that $r+s$ and $r+s+n$ are possible return times to i. Indeed, notice that

$$P_{ii}^{r+s} = P(\overline{X}_{r+s} = i \mid \overline{X}_0 = i)$$

$$\geq P(\overline{X}_{r+s} = i, \overline{X}_r = j \mid \overline{X}_0 = i)$$

$$= P(\overline{X}_{r+s} = i \mid \overline{X}_r = j, \overline{X}_0 = i) P(\overline{X}_r = j \mid \overline{X}_0 = i)$$

$$= P_{ji}^s P_{ij}^r > 0$$

and

$$P_{ii}^{r+s+n} = P(\overline{X}_{r+s+n} = i \mid \overline{X}_0 = i)$$

$$\geq P(\overline{X}_{r+s+n} = i, \overline{X}_{r+n} = j, \overline{X}_r = j \mid \overline{X}_0 = i)$$

$$= P_{ij}^r \cdot P_{jj}^n P_{ji}^s > 0.$$

Therefore $d(i)$ divides $r+s$ and $r+s+n$. That is, there exist $k, \ell \in \mathbb{Z}$ such that $d(i)k = r+s$ and $d(i)\ell = r+s+n$. This implies

$$d(i)k - d(i)\ell = n \Rightarrow d(i)(k-\ell) = n$$

which implies $d(i)$ divides n. Therefore $d(i) \leq d(j)$. The same argument with the roles of i and j reversed shows that $d(j) \leq d(i)$.