Introduction to Brownian Motion

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Background

- Einstein showed that the position x of a particle at time t was described by the partial differential equation,

$$\frac{\partial}{\partial t} f(x,t) = \frac{1}{2} \frac{\partial^2}{\partial x^2} f(x,t),$$

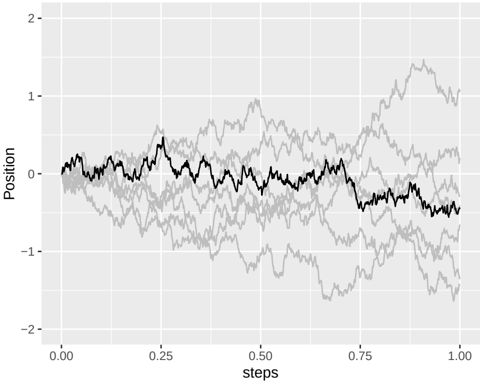
  where $f(x,t)$ is the number of particles per unit volume at position x and time t.
- The solution to the equation above is

$$f(x,t) = \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t},$$

  which is the probability density function of the normal distribution with mean 0 and variance t.

Brownian Motion Properties

1. $B_0 = 0$
2. Normal Distribution
3. Continuous Paths
4. Stationary Increments
5. Independent Increments

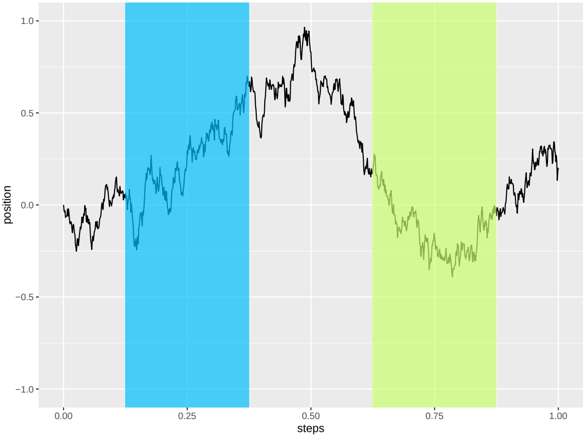


Initial Position and Normal Distribution

- Let B_t be a standard Brownian motion random variable
  - $B_t = 0$ for $t = 0$
- $B_t \sim \mathcal{N}(0, t)$
- The function $t \mapsto B_t$ is continuous, with probability 1.

Stationary and Independent Increments

- For $s, t > 0$ the distribution of $B_{t+s} - B_s$ does not depend on s
  - That is, $B_{t+s} - B_s, B_t \sim \mathcal{N}(0, t)$.
- If $0 \leq q < r \leq s < t$, then $B_t - B_s$ and $B_r - B_q$ are independent random variables



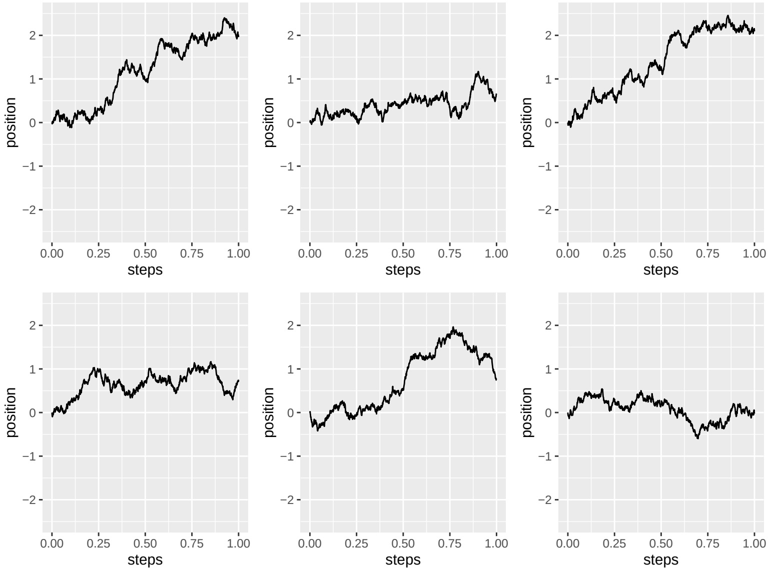
Simulating Brownian Motion

R: Simulating Brownian Motion

```
# bm.R
> n <- 1000
> t <- 1
> bm <- c(0, cumsum(rnorm(n,0,sqrt(t/n))))
> steps <- seq(0,t,length=n+1)
> plot(steps,bm,type="l")
```

- **t** is the distance to travel
- **n** is the number of random variables we are selecting according to a normal distribution with mean **0** and standard deviation **sqrt(t/n)**
- **bm** starts at **0** and ends at the cumulative sum of each of the Brownian motion random variables

Simulating Brownian Motion



Brownian Motion and Random Walk

Discrete-time, discrete-state random walk:

- Let X_i be i.i.d variables taking on values ± 1 with $p = 1/2$ for each.
- Let $S_0 = 0$ and for integer $t > 0$, let $S_t = X_1 + X_2 + \cdots + X_t$.
- $E(S_t) = 0$ and $Var(S_t) = t$
- For large t, S_t is approximately normally distributed.

Brownian Motion and Random Walk

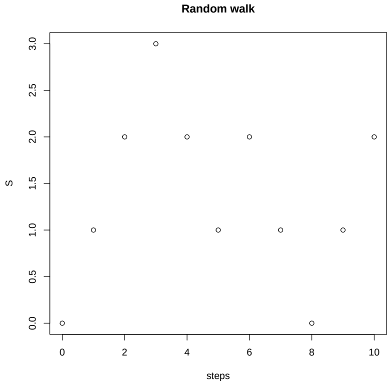


Figure: $S_t, t = 0, 1, \ldots, 10$

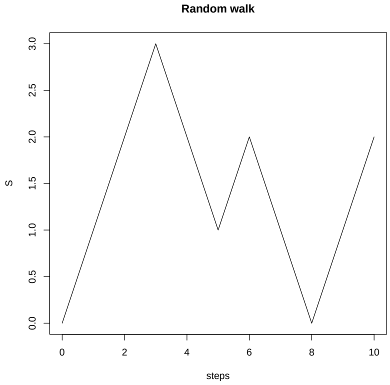


Figure: $S_t, 0 \leq t \leq 10$

Brownian Motion and Random Walk

Continuous-time, continuous-state random walk:

- Extending the definition of S_t to real number $t \geq 0$, let

$$S_t = \begin{cases} X_1 + X_2 + \cdots + X_t, & \text{if } t \text{ is an integer,} \\ S_{\lfloor t \rfloor} + X_{\lfloor t \rfloor + 1}(t - \lfloor t \rfloor), & \text{otherwise.} \end{cases}$$

- In this case, $E(S_t) = 0$ and $Var(S_t) \approx t$.

$$\begin{aligned} Var(S_t) &= Var(S_{\lfloor t \rfloor} + X_{\lfloor t \rfloor + 1}(t - \lfloor t \rfloor)) \\ &= Var(S_{\lfloor t \rfloor}) + (t - \lfloor t \rfloor)^2 Var(X_{\lfloor t \rfloor + 1}) \\ &= \lfloor t \rfloor + (t - \lfloor t \rfloor)^2 \\ &\approx t. \end{aligned}$$

Scaling to Construct Brownian Motion

- Let n be an integer and $S_t^{(n)} = S_{nt}/\sqrt{n}$. On any interval of t, the new process has n times as many steps as the original walk.

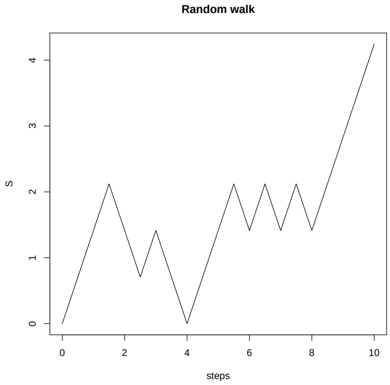


Figure: $S_t^{(2)}, 0 \leq t \leq 10$

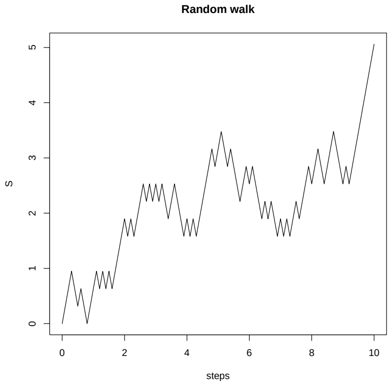


Figure: $S_t^{(10)}, 0 \leq t \leq 10$

Scaling to Construct Brownian Motion

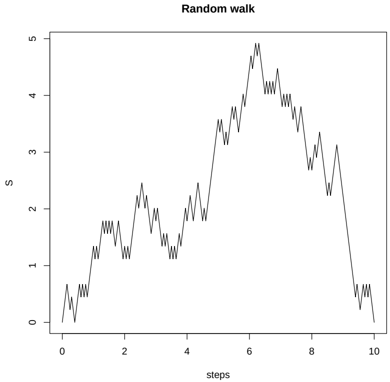


Figure: $S_t^{(20)}, 0 \leq t \leq 10$

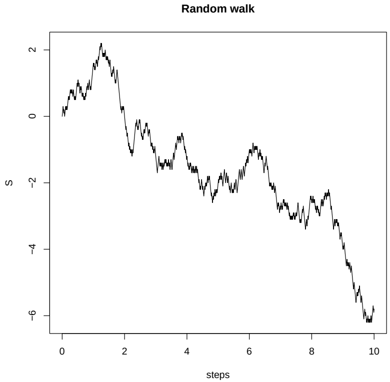


Figure: $S_t^{(100)}, 0 \leq t \leq 10$

Scaling to Construct Brownian Motion

The Central Limit Theorem: Let $X_1, X_2, \dots$ be an i.i.d sequence of random variables with finite mean μ and variance σ^2. For $n = 1, 2, \dots$, let $S_n = X_1 + \cdots + X_n$. Then as $n \to \infty$,

$$\frac{S_n/n}{\sigma/\sqrt{n}} \sim \mathscr{N}(0,1),$$

which implies that

$$\frac{S_n}{\sqrt{n}} \sim \mathscr{N}(0,1).$$

Invariance Principle

- The construction of Brownian motion from simple symmetric random walk can be generalized so that we start with any i.i.d. sequence $X_1, \ldots, X_n$ with mean 0 and variance 1.
- *Donsker's invariance principle*: Let $S_n = X_1 + \cdots + X_n$. Then, $S_{nt}/\sqrt{n}$ converges to B_t as $n \to \infty$.

Thanks for listening!