On Caterpillars, Trees, and Stochastic Processes

Camille Cheng and Giovanna Giuliano

Introduction to the problem

“A bird carrying a caterpillar in its beak flies over a tree. The caterpillar jerks free, and falls into the tree onto a node. What is the expected number of leaves the caterpillar finds above her in the portion of tree stemming from her landing point?”

Why is this a stochastic problem?

What must we specify to be able to solve this problem?

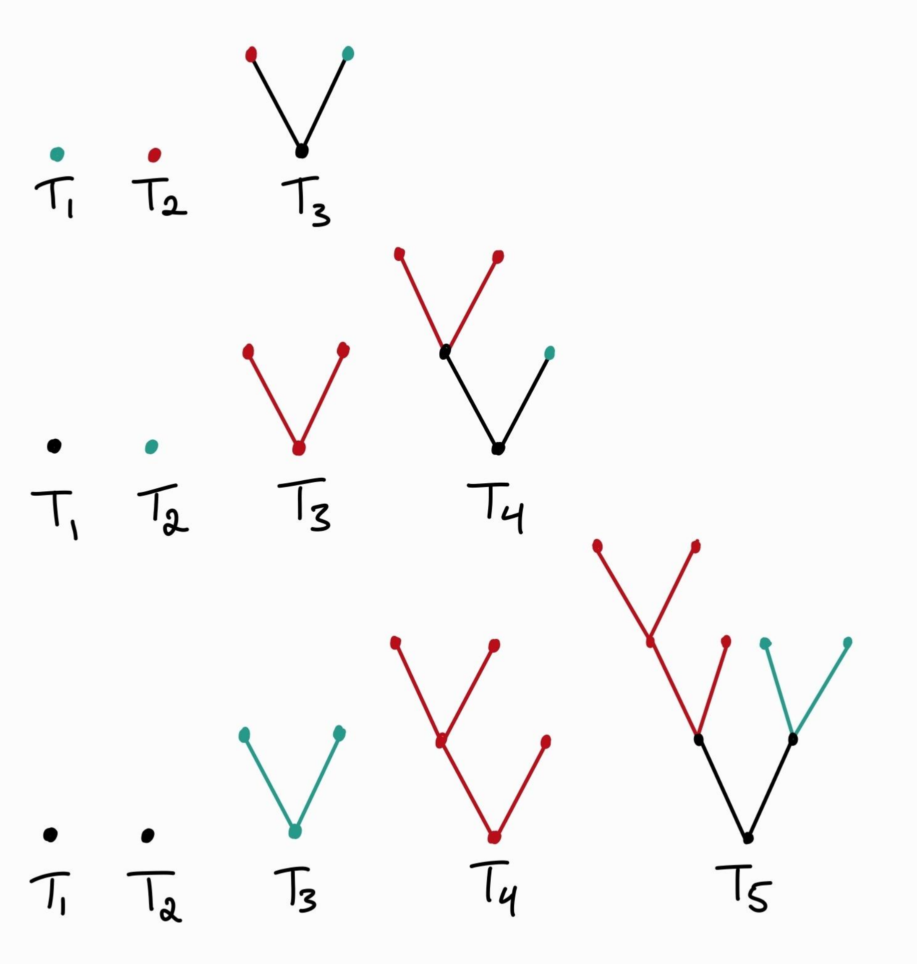
Problem specifications

Define a sequence of Fibonacci trees: each new tree T_k is built by adding the previous two trees onto it. The number of "leaf-nodes," or the number of vertices at the ends, will follow the Fibonacci sequence for k >= 3.

Let F_k be the number of leaf-nodes in T_k.

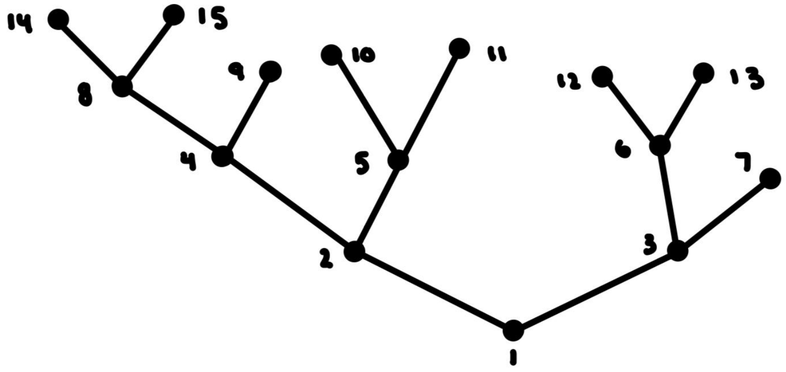
Let N_k be the total number of nodes in T_k. It can be shown that $N_k = 2F_k - 1$.

The caterpillar falls onto a node with equal probability (if there are N_k nodes, then the probability any one node is landed on is $1/N_k$)



Solving the problem: T_6

Let's consider answering this problem for a simple tree, T_6



For each node 1-15, we can count the number of leaves above that node.

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
8	5	3	3	2	2	0	2	0	0	0	0	0	0	0

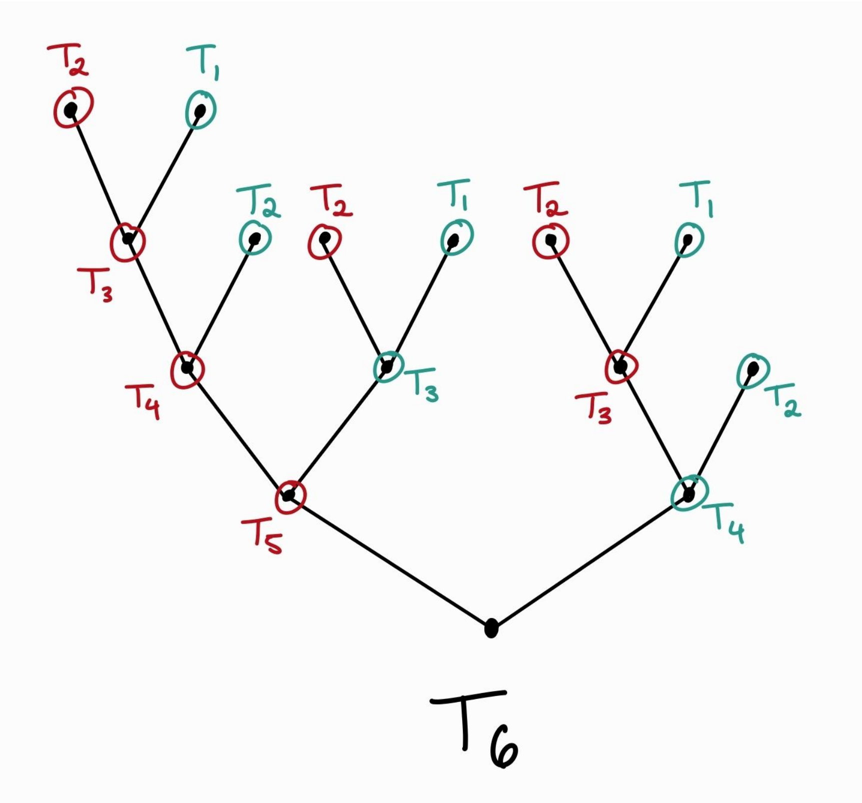
Let L be the number of leaves above the node. Based on our table above, we can calculate the probability of each value of L

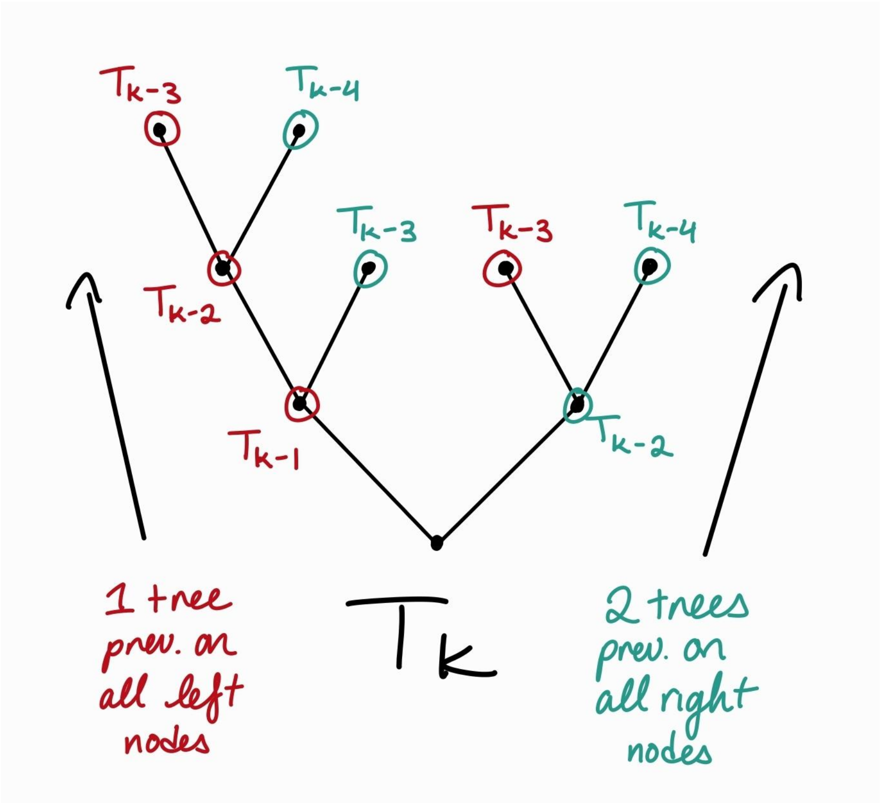
L	0	2	3	5	8
P(L)	8/15	3/15	2/15	1/15	1/15

And we can now calculate the expectation of L using our definition of expectation:

$$E[L] = \sum_{n=1}^{\infty} \text{L}p(L) = 0 \cdot \frac{8}{15} + 2 \cdot \frac{3}{15} + 3 \cdot \frac{2}{15} + 4 \cdot \frac{1}{15} + 5 \cdot \frac{1}{15} = \frac{5}{3}$$

A general solution



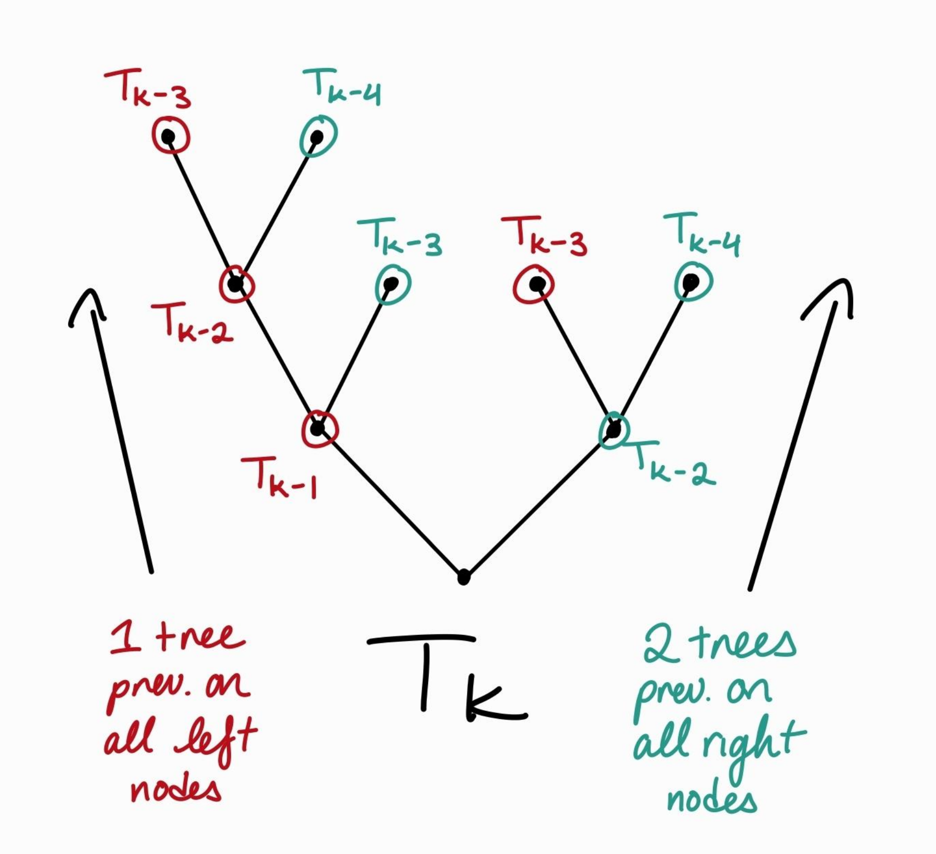


A general solution

Since each node is the base of a particular Fibonacci tree, the number of leaf-nodes above it will simply be F_k for that sub-tree.

Note two things:

- L=0 will occur F_k times as each leaf-node in tree T_k has 0 nodes above it.
- Any other value $L=F_r$ where r = 3,..,k will be F_{k-r+1}.



A general solution

With this, we can define the following:

$$\mu_L = \frac{2F_{k-2} + 3F_{k-3} + 5F_{k-4} + ... + F_k \cdot 1}{N_k} = \frac{\sum_{r=3}^{k} F_r F_{k-r+1}}{2F_k - 1} = \frac{s_k}{2F_k - 1}$$

And we can find the polynomial generating function for s_k

$$G(x) = \sum_{k=0}^{\infty} s_k x^k = \frac{x^2(2+x)}{(1-x-x^2)^2}$$

What are generating functions?

Lets consider a simpler example, the fibonacci sequence, to understand generating functions.

Consider $0, 1, 1, 2, 3, 5, 8, ...$
$F_k = F_{k-1} + F_{k-2}, F_0 = 0, F_1 = 1, k \geq 2$

$$F_k - F_{k-1} - F_{k-2} = 0$$

Now let $H(x) = \sum_{k=0}^{\infty} F_k x^k$ be the generating function of the Fibonacci sequence.

$$H(x) = F_0 + F_1 x + F_2 x^2 + F_3 x^3 + F_4 x^4 + ...$$

$$-xH(x) = -F_0 x - F_1 x^2 - F_2 x^3 - F_3 x^4 + ...$$

$$-x^2 H(x) = -F_0 x^2 - F_1 x^3 - F_2 x^4 + ...$$

So you end up with $H(x)(1 - x - x^2) = F_0 + (F_1 - F_0)x = x$

$$
\begin{aligned}
H(x) &= \frac{x}{1-x-x^2} \\
&= \frac{x}{(x-\frac{-1+\sqrt{5}}{2})(x-\frac{-1-\sqrt{5}}{2}} \\
&= \frac{A}{x-\alpha}+\frac{B}{x-\beta} \\
&= \frac{\frac{A}{\alpha}}{\frac{x}{\alpha}-1}+\frac{\frac{B}{\beta}}{\frac{x}{\beta}-1} \\
&= \frac{-\frac{A}{\alpha}}{1-\frac{x}{\alpha}}+\frac{-\frac{B}{\beta}}{1-\frac{x}{\beta}} \\
&= \frac{-A}{\alpha}\sum_{k=0}^{\infty}(\frac{x}{\alpha})^k-\frac{-B}{\beta}\sum_{k=0}^{\infty}(\frac{x}{\beta})^k \\
&= \sum_{k=0}^{\infty}\frac{-A}{\alpha^{k+1}}x^k-\sum_{k=0}^{\infty}\frac{-B}{\beta^{k+1}}x^k \\
&= \sum_{k=0}^{\infty}(\frac{-A}{\alpha^{k+1}}+\frac{-B}{\beta^{k+1}})x^k
\end{aligned}
$$

So $F_k = \frac{-A}{\alpha^{k+1}} + \frac{-B}{\beta^{k+1}}$

Back to our problem…

So how does this connect to the generating function from our problem?

Given G(x), we can find the power series for this polynomial using technology like WolframAlpha:

$$\sum_{n=0}^{\infty} \frac{1}{25} x^n \left(2^{-1-2n} (1+\sqrt{5})^n \left(-8\sqrt{5}\left(2^n - (-3+\sqrt{5})^n\right) - 5(-3+\sqrt{5})^{1+n} n + 5 \times 2^n (3+\sqrt{5}) n\right)\right)$$

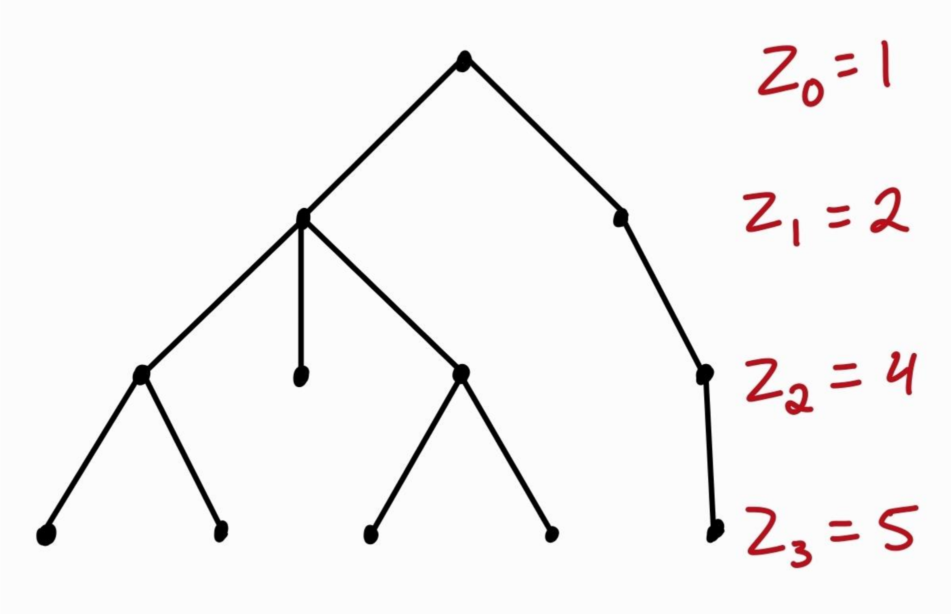
A different type of branching process

Imagine a population of individuals, where each person independently has k children in accordance with a probability distribution

a = $(a_0, a_1, a_2, \ldots)$.

Let Z_n be the size of the n^{th} generation for n >=0. If $Z_0 = 1$, then the sequence $Z_0, Z_1, \ldots$ is a branching process.

Because the size of each successive generation Z_n depends only on the size of the previous generation Z_{n-1}, this branching process is a Markov chain.



References

- “On Caterpillars, Trees, and Stochastic Processes” by John C. Turner
- Chapter 4 of our class textbook