A Real-World Markov Chain Arising in Recreational Volleyball

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Overview

- The paper authored by David J. Aldous and Madelyn Cruz presents a Markov chain model to understand the dynamics of team composition in recreational volleyball.

- This research is motivated by the practical challenge of mixing players into teams such that the team compositions change from game to game. It aims to offer a realistic model that captures the nuances of player movement and team dynamics over successive games.

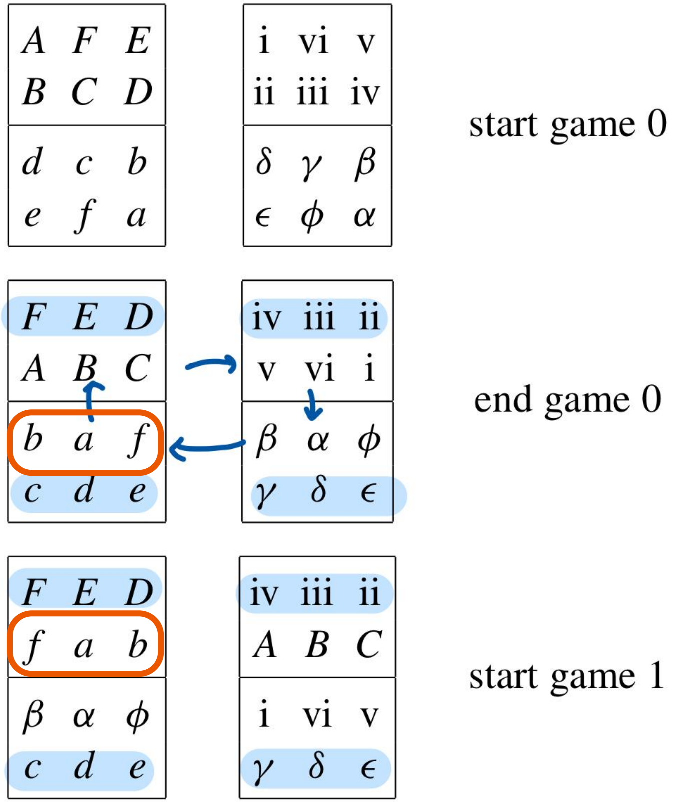
Table of contents

Introduction and Model

Settings: 24 players -> 2 courts -> each with 2 team with 6 players on a half court. 7 or 8 successive games in 2-hour period.

Rotations: At the end of one stage, the players in the back row of each team stay in these positions for the start of the next game, while the front row players move (clockwise in the gym) to the same positions in the next quadrant.

State Space: {24!} (assignment to players to positions)



Proofs

Irreducibility and Aperiodicity

Mixing Efficiency

The main focus of this paper is to determine how effectively the described protocol mixes the players. The analysis reveals that the scheme tends to evenly distribute players among teams over several games, thus achieving a good level of mixing as evidenced by the properties of the Markov chain being doubly stochastic.

Irreducibility

Notation

1. Each of the four quadrants (half-courts) is labeled (A, B, C, D)
2. $0 \leq x_i \leq 5$ indicates the number of positions (modulo 6) rotated by the team in the relevant quadrant

$$A^{x_1} C^{x_2} B^{x_3} D^{x_4}$$

3. Symbol E : the final movement (the front row players in each quadrant move to the same positions in the next quadrant

Note: EEEE would code the identity move (where it remain in original state)

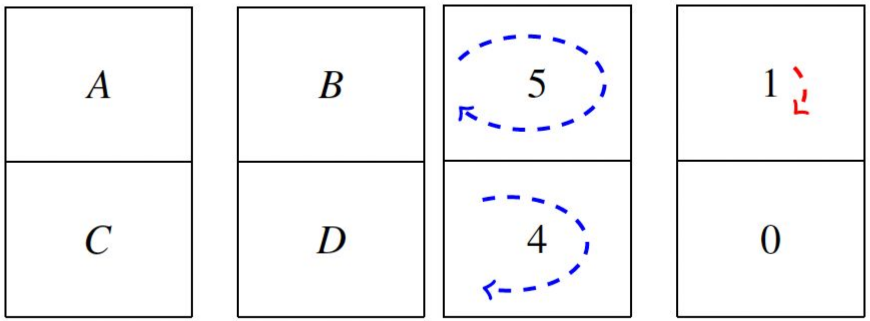


Figure 2. Left: labeling of the four quadrants. Right: rotations involved in step A^5C^4BE.

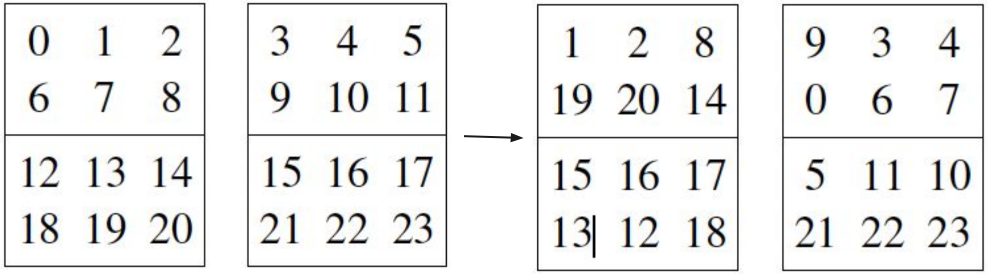


Figure 3. The effect of step A^5C^4BE.

High-level description

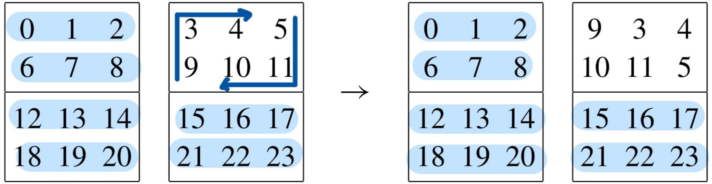
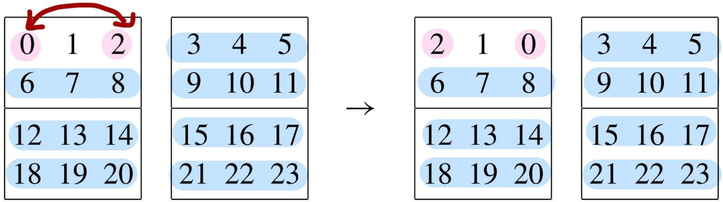
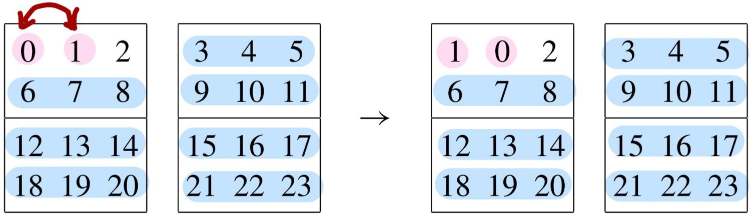


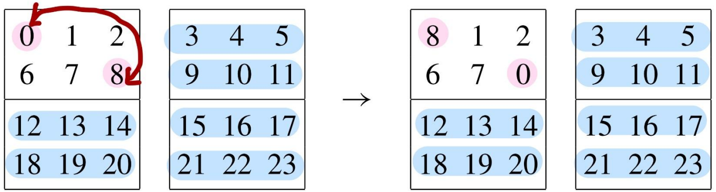
Figure 4. The effect of sequence $BEEEE$.



(b) Transpose two players in the same quadrant with one space in between (G).



(a) Transpose two adjacent players in the same quadrant (sequence F).



(c) Transpose two players in the same quadrant with two spaces in between (H).

"Random adjacent transposition" shuffling scheme: Transpositions achieved by the specific sequences F, G, H

High-level description

$$X := AE\ B^2D^3EEE\ A^2C^3E\ B^3D^3EEE\ A^5E\ B^5EEE\ AE\ B^5EEE\ AC^3$$

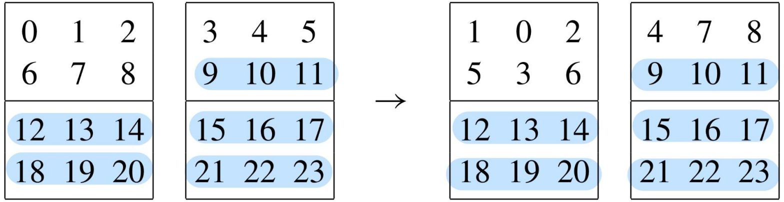


Figure 6. The effect of sequence X.

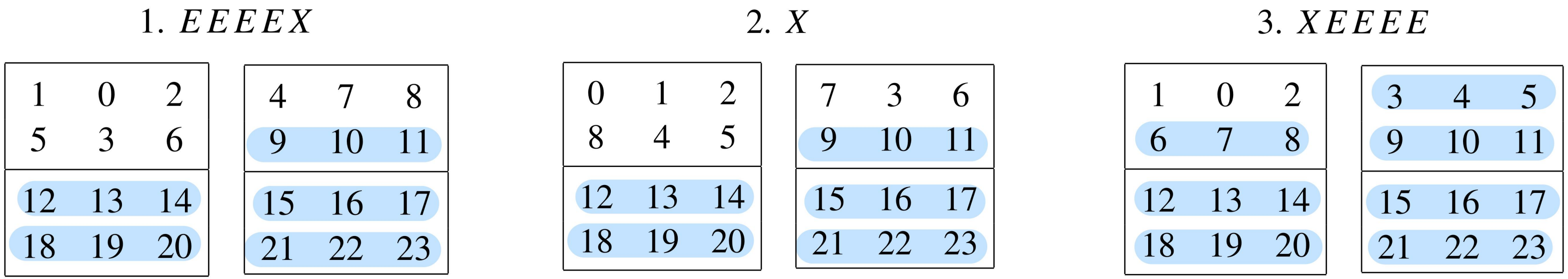


Figure 8. Step-by-step trajectory of sequence F.

Aperiodicity

Self-loops:

i) We know from previous sequence : we can move from one state to that same state in **four steps**—E E E E

ii) It’s also possible to remain in the same state in the transition matrix for some **odd n**.
(There are 163 E’s in the unexpanded notation, and 68, 424, and 340 steps from the one F, two G’s, andone H, respectively, so there are 995 steps in this sequence.)

1–13. *AEEEE AEAE CE DEEEE DEDE* (move to a certain other arrangement)

14	7	6	3	4	5
0	21	22	8	2	1
23	15	16	18	11	10
19	20	9	13	12	17

14–32. *EEDEEEE DEEEE DEE CEEEE CEEE D* (fixing back row of D)

14	7	6	3	4	5
8	2	1	10	11	18
17	12	13	0	20	19
9	16	15	21	22	23

33–57. *CEEEE CEEEE CEEE AEEEE AEEE CEEE CEEEE C* (fixing back row of C)

11	10	14	3	4	5
9	16	15	8	2	1
17	12	0	13	7	6
18	19	20	21	22	23

58–67. *EAEEEE AEBEE AE BBBBBE* (fixing (17,16,15) using Procedure B then some migration)

9	12	0	3	4	11
6	7	13	14	10	5
1	2	8	15	16	17
18	19	20	21	22	23

68–87. *BAEEEE AEBEEE AEBE BEEEE BEEEE BEE* (fixing (14,13,12) using Procedure B then some migration)

0	7	6	1	2	8
4	9	3	10	5	11
12	13	14	15	16	17
18	19	20	21	22	23

88–99. *BEEEE BEBEEE AEBEEE* (fixing (5,4,3) using Procedure B)

9	0	7	3	4	5
1	6	10	11	8	2
12	13	14	15	16	17
18	19	20	21	22	23

100–135. *AEEEE AEEEE AGAEEEE AEEEE AEEEE AEEE AEEEE AEEEE AEEEE AE* (fixing (9,10,11) using Steps 15 and 16 of the algorithm)

0	2	8	3	4	5
6	1	7	9	10	11
12	13	14	15	16	17
18	19	20	21	22	23

136–163. *AEEEE AHA AGA AEEEE AEEEE AEEEE AFAEEEE AEEEE AEEEE* (fixing A)

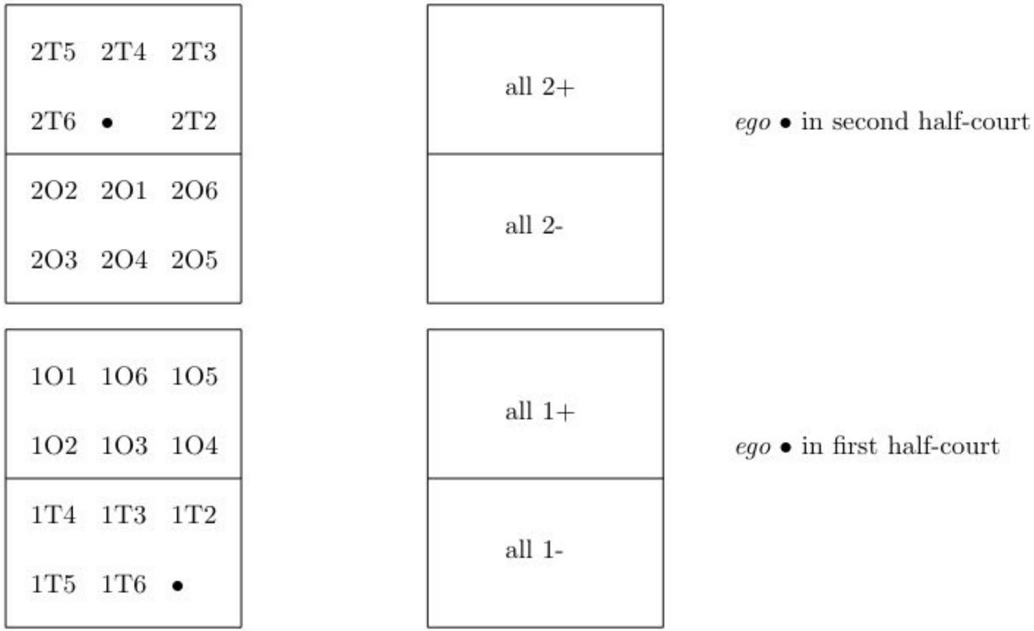
0	1	2	3	4	5
6	1	7	9	10	11
12	13	14	15	16	17
18	19	20	21	22	23

The friend chain

Describe the state:

★ **The quadrant *ego* is in**
  ○ 1 ~ quadrant C
  ○ 2 ~ quadrant A
★ **The position of *friend* relative to *ego***
  ○ T: In the same team
  ○ O: In the current opposing team
  ○ On the other court: 1+, 1-, 2+, 2-
★ ***friend*'s position:** 1, 2, 3, 4, 5, or 6, counterclockwise from *ego* or *ego** (opponent in the same position as ego)

A **26-state** chain indicating the relative position of *ego* and *friend* (by symmetry)
The process is invariant under a **half-turn**.



Transition matrix for the friend chain

The stationary distribution π is induced from the uniform stationary distribution of the big chain:

★ probability **6/46** for the states 1+, 1−, 2+, 2−

★ probability **1/46** for the 22 remaining states.

	1+	1−	1T2	1T3	1T4	1T5	1T6	1O1	1O2	1O3	1O4	1O5	1O6	2T2	2T3	2T4	2T5	2T6	2O1	2O2	2O3	2O4	2O5	2O6	2+	2−
1+	1/4	1/4																							1/4	1/4
1−		1/4	1/36	2/36	3/36	2/36	1/36												3/36	2/36	1/36		1/36	2/36		1/4
1T2			1/3						1/6					1/3						1/6						
1T3				1/6						1/3					1/6						1/3					
1T4											1/2											1/2				
1T5						1/6						1/3					1/6						1/3			
1T6							1/3						1/6					1/3						1/6		
1O1	1/12							1/4	1/12				1/12	1/24				1/24							$1\frac{0}{24}$	
1O2	1/6							3/24	1/6	1/24				1/12	1/12										1/3	
1O3	1/3								1/12	1/12				1/24	1/6	1/8									1/6	
1O4	5/12									1/24		1/24			1/12	3/12	1/12								1/12	
1O5	1/3											1/12	1/12			3/24	1/6	1/24							1/6	
1O6	1/6							3/24				1/24	1/6				1/12	1/12							1/3	
2T2		1/6	1/3											1/3											1/6	
2T3		1/3		1/6											1/6										1/3	
2T4		1/2																							1/2	
2T5		1/3				1/6											1/6								1/3	
2T6		1/6					1/3											1/3							1/6	
2O1	1/12	5/12												1/24				1/24	1/4	1/12				1/12		
2O2	1/6	1/3												1/12	1/12				3/24	1/6	1/24					
2O3	1/3	1/6												1/24	1/6	3/24				1/12	1/12					
2O4	5/12	1/12													1/12	3/12	1/12				1/24		1/24			
2O5	1/3	1/6														3/24	1/6	1/24					1/12	1/12		
2O6	1/6	1/3															1/12	1/12	3/24				1/24	1/6		
2+			1/36	2/36	3/36	2/36	1/36	3/36	2/36	1/36		1/36	2/36												1/4	1/4
2−	1/4								1/36	2/36	3/36	2/36	1/36							1/36	2/36	3/36	2/36	1/36		1/4

Numerics for the friend chain

Standard theory quantifies “closeness to stationarity after n steps” via ***variation distance $d^*(n)$*** or ***separation distance $s^*(n)$*** from worst-case start.

$$d^*(n) := \max_i \frac{1}{2} \sum_j |p_{ij}^n - \pi_j|,$$

$$s^*(n) := \max_{i,j}(1 - p_{ij}^n / \pi_j)$$

Another measure of distance to stationarity is the $\boldsymbol{L^2}$ or the $\boldsymbol{\chi^2}$ distance. The $\boldsymbol{L^2}$ distance between P_i and the stationary distribution π after *n* steps is

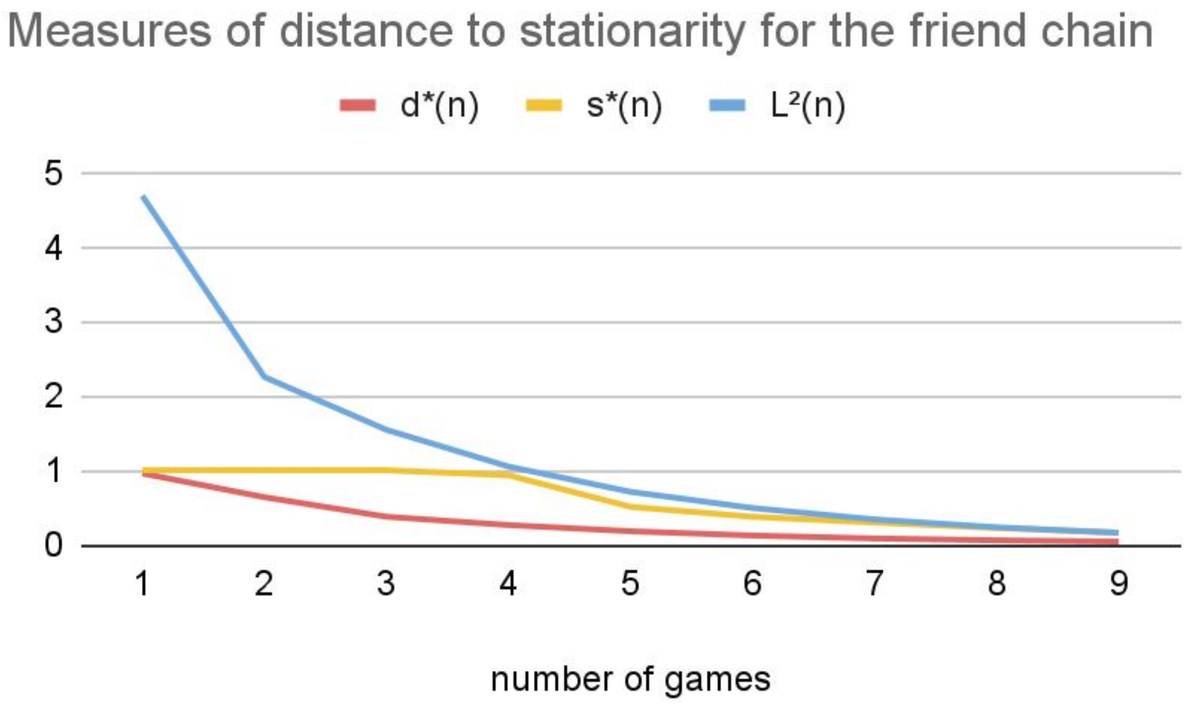
$$\| P^n(i,\cdot) - \pi \|_2 = \sqrt{\sum_j \frac{(p_{ij}^n - \pi_j)^2}{\pi_j}}$$

$$\| P^n - \pi \|_2 = \max_i \| P^n(i,\cdot) - \pi \|_2 .$$

Numerics for the friend chain

n	1	2	3	4	5	6	7	8	9
$d^*(n)$	0.957	0.638	0.375	0.263	0.180	0.122	0.083	0.058	0.040
$s^*(n)$	1	1	1	0.933	0.508	0.374	0.297	0.223	0.160
$L^2(n)$	4.690	2.254	1.544	1.05	0.71	0.492	0.339	0.233	0.159

Table 1. Measures of distance to stationarity for the friend chain, after n games.


Measures of distance to stationarity for the friend chain
d*(n)
s*(n)
L²(n)
5
4
3
2
1
0
1
2
3
4
5
6
7
8
9
number of games

Numerics

If *friend* is playing with or against *ego*, the mean number of games in which *friend* is on the opposite team (or the same team) can be computed depending on the distance from friend's initial position to *ego* or *ego**'s initial position.

start	1+	1−	$1T2$	$1T3$	$1T4$	$1T5$	$1T6$	$1O1$	$1O2$	$1O3$	$1O4$	$1O5$	$1O6$
OT	1.607	1.962	1.803	2.107	2.222	2.107	1.803	3.059	2.894	2.606	2.482	2.606	2.894
ST	1.093	1.515	3.773	2.725	2.314	2.725	3.773	1.421	1.499	1.550	1.523	1.550	1.499

Table 2. Mean number of games (out of eight) in which *friend* is on the opposite team OT (and the same team ST) as *ego*, who starts in the first quadrant.

start	$2T2$	$2T3$	$2T4$	$2T5$	$2T6$	$2O1$	$2O2$	$2O3$	$2O4$	$2O5$	$2O6$	2+	2−
OT	1.493	1.678	1.700	1.678	1.493	3.059	2.894	2.606	2.482	2.606	2.894	1.962	1.940
ST	3.778	2.698	2.297	2.698	3.778	1.421	1.499	1.550	1.523	1.550	1.499	1.515	1.103

Table 3. Mean number of games (out of eight) in which *friend* is on the opposite team OT (and the same team ST) as *ego*, who starts in the second quadrant.

Fundamental matrix of the friend chain

Consider an absorbing Markov chain on k states for which t states are transient and k − t states are absorbing. The canonical form of transition matrix P is:

$$P = \left(\begin{array}{c|c} Q & R \\ \hline \mathbf{0} & I \end{array}\right)$$

Dimension:

- ★ Q is a $t \times t$ matrix
- ★ R is a $t \times (k - t)$ matrix
- ★ $\mathbf{0}$ is a $(k - t) \times t$ matrix of 0s
- ★ I is the $(k - t)\times(k - t)$ identity matrix.

The **fundamental matrix F** determines the mean time to go from one given initial state to another given target state.

$$F = (I - Q)^{-1}.$$

Fundamental matrix of the friend chain

We can pick the target state(s) and make them into absorbing state(s).

Suppose ***ego***'s position is randomized in the left court. We will calculate how long it takes for ***friend*** to be right next to ***ego*** in quadrant 1.

That is, **either 1T2 or 1T6 is an absorbing state**.

In this case, the **fundamental matrix** of the absorbing chain is a **25 x 25 matrix.**

Fundamental matrix

We can observe that there is a symmetry property in the number of games which confirms the claim that the number of games depends on *friend*'s shortest distance to *ego*.

1+	1-	1T3	1T4	1T5	1T6	101	102	103	104	105	106	2T2
93.14581	90.77034	92.38001	94.42458	94.62757	95.29642	90.09790	88.11483	90.90195	93.42458	93.59902	93.05945	46.88517

2T3	2T4	2T5	2T6	201	202	203	204	205	206	2+	2-
92.29228	91.77034	92.74179	94.53338	90.09790	88.11483	90.90195	93.42458	93.59902	93.05945	90.77034	93.89675

1+	1-	1T2	1T3	1T4	1T5	101	102	103	104	105	106	2T2
93.14581	90.77034	95.29642	94.62757	94.42458	92.38001	90.09790	93.05945	93.59902	93.42458	90.90195	88.11483	94.53338

2T3	2T4	2T5	2T6	201	202	203	204	205	206	2+	2-
92.74179	91.77034	92.29228	46.88517	90.09790	93.05945	93.59902	93.42458	90.90195	88.11483	90.77034	93.89675

Final Remarks

Card Shuffling Analogy

“7 shuffles are necessary and suffice to approximately randomize 52 cards” - Diaconis

Thank you.