§4.1-4.2 Expectation

Def Let X be a discrete random variable with range $S \subseteq \mathbb{R}$. The probability mass function of X is a function $m: S \to [0,1]$ where $m(k) = P(X = k)$ for each $k \in S$. The expectation of X (also called the expected value of X or mean of X) is $E[X] = \sum_{k \in S} k\, m(k)$.

Remark ① $E[X]$ can be thought of as a weighted average of the possible values of X, where the weights are the values given by their probabilities (i.e. their probability mass).

② In analogy with physics we can think of $E[X]$ as the center of mass of the distribution.

③ Later we will learn that if $X_1, X_2, \ldots$ is an i.i.d. sequence with the same distribution as X, then

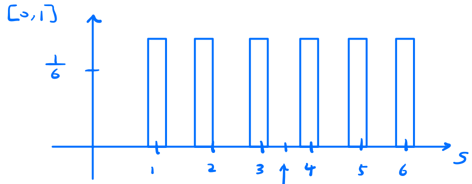
$$\lim_{n \to \infty} \frac{X_1 + \cdots + X_n}{n} = E[X] \quad \text{with probability } 1.$$

This is called the Strong Law of Large Numbers.

Example Let $X \sim \mathrm{Unif}(\{1,2,3,4,5,6\})$. Find $E[X]$.

$$E[X] = \sum_{k=1}^{6} k \cdot P(X = k) = \sum_{k=1}^{6} k \cdot \frac{1}{6}$$

$$= \frac{1}{6}(1+2+3+4+5+6) = 3.5$$



$E[X]$ is the center of mass or "balancing point"

<u>Example</u> if $X \sim \mathrm{Unif}\{1,2,\dots,n\}$, then $E[X] = \frac{n+1}{2}$

<u>Proof</u>
$$E[X] = \sum_{k=1}^{n} k\, P(X=k)$$
$$= \frac{1}{n} \sum_{k=1}^{n} k$$
$$= \frac{1}{n} \frac{n(n+1)}{2} = \frac{n+1}{2}.$$

<u>Example</u> if $X \sim \mathrm{Pois}(\lambda)$, then $E[X] = \lambda$.

<u>Proof</u>
$$E[X] = \sum_{k=0}^{\infty} k\, P(X=k)$$
$$= \sum_{k=0}^{\infty} k \cdot e^{-\lambda} \frac{\lambda^k}{k!}$$
$$= \sum_{k=1}^{\infty} k e^{-\lambda} \frac{\lambda^k}{k!} \quad \text{(the } k=0 \text{ term was } 0\text{)}$$
$$= e^{-\lambda} \sum_{k=1}^{\infty} \frac{\lambda^k}{(k-1)!} \quad \text{(cancel with factorial)}$$
$$= \lambda e^{-\lambda} \sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!} \quad \text{(factor out } \lambda^1\text{)}$$
$$= \lambda e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \quad \text{(re-index)}$$
$$= \lambda e^{-\lambda} \cdot e^{\lambda} \quad \left(\text{Taylor series for } e^x \text{ is } \sum_{k=0}^{\infty} \frac{x^k}{k!}\right)$$
$$= \lambda$$

Sometimes called the Law of the Unconscious Statistician

Theorem If X is a discrete random variable whose range is S and $f : S \to \mathbb{R}$ is a given function then

$$E[f(X)] = \sum_{k \in S} f(k) P(X = k).$$

Example If the radius of a sphere is $R \sim Unif(\{1,2,3,4,5,6\})$ find the expected value of its volume.

The volume is the random variable $V = \frac{4}{3}\pi R^3$, so

$$E[V] = \sum_{k=1}^{6} \frac{4}{3}\pi k^3 \cdot P(X = k)$$

$$= \frac{4}{3}\pi \left(1^3 + 2^3 + 3^3 + 4^3 + 5^3 + 6^3\right) \cdot \frac{1}{6}$$

$$= 98\pi$$

Warning In general, $E[f(X)] \neq f(E[X])$.

However...

Theorem Let $a, b \in \mathbb{R}$ be given constants. Then

$$E[aX + b] = aE[X] + b.$$

Proof

$$E[aX + b] = \sum_{k \in S} (ak + b) P(X = k)$$

$$= \sum_{k \in S} ak\, P(X = k) + \sum_{k \in S} b\, P(X = k)$$

$$= a \sum_{k \in S} k\, P(X = k) + b \sum_{k \in S} P(X = k)$$

$$= aE[X] + b.$$

Problem 1. Suppose a person always runs a mile in either 4 minutes, 5 minutes, 6 minutes, 10 minutes, or 15 minutes, uniformly at random. Let T denote the time of their run, in minutes, on a given day.

a. Find $E[T]$.

b. Let S be their speed in miles per minute.

1. Express S in terms of T.
2. Find $E[S]$ using the Law of the Unconscious Statistician.

(a) $$E[T] = 4\left(\tfrac{1}{5}\right) + 5\left(\tfrac{1}{5}\right) + 6\left(\tfrac{1}{5}\right) + 10\left(\tfrac{1}{5}\right) + 15\left(\tfrac{1}{5}\right)$$

$$= (4+5+6+10+15)\left(\tfrac{1}{5}\right) = 8 \text{ minutes}$$

(b) $$S = \frac{1}{T}, \text{ so } E[S] = \frac{1}{4}\left(\tfrac{1}{5}\right) + \frac{1}{5}\left(\tfrac{1}{5}\right) + \frac{1}{6}\left(\tfrac{1}{5}\right) + \frac{1}{10}\left(\tfrac{1}{5}\right) + \frac{1}{15}\left(\tfrac{1}{5}\right)$$

$$= \left(\frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{10} + \frac{1}{15}\right)\left(\tfrac{1}{5}\right) = \frac{47}{300}$$

Problem 3. Consider the following gambling game, which costs \$7 up front to play. You toss a coin 5 times. If the coin comes up heads fewer than 3 times, you get nothing back. If the coin comes up heads 3 times you get your money back. The coin comes up heads 4 times, you get \$10 back. If the coin comes up heads 5 times, you get \$50 back. Let W represent your net winnings.

a. Find the range of W.

b. Find the probability mass function of W.

c. Find $E[W]$.

(a) $\{-7, 0, 3, 43\}$

(b) Let $X \sim \text{Bin}\left(5, \tfrac{1}{2}\right)$. Then

$$P(W=-7) = P(X \leq 2) = \left(\tfrac{1}{2}\right)^5 + 5\left(\tfrac{1}{2}\right)^5 + \binom{5}{2}\left(\tfrac{1}{2}\right)^5 = 16\left(\tfrac{1}{2}\right)^5 = \tfrac{1}{2}$$

$$P(W=0) = P(X=3) = \binom{5}{3}\left(\tfrac{1}{2}\right)^5 = \frac{10}{32} = \frac{5}{16}$$

$$P(W=3) = P(X=4) = \binom{5}{4}\left(\tfrac{1}{2}\right)^5 = \frac{5}{32}$$

$$P(W=43) = P(X=5) = \left(\tfrac{1}{2}\right)^5 = \frac{1}{32}$$

(c) $$E[W] = -7\left(\tfrac{1}{2}\right) + 0\left(\tfrac{5}{16}\right) + 3\left(\tfrac{5}{32}\right) + 43\left(\tfrac{1}{32}\right) = -1.6875$$

Problem 2. Let $p \in (0,1)$ and suppose $X \sim \text{Ber}(p)$. Recall that this means X is a random variable whose range is $\{0,1\}$ and $P(X=1)=p, P(X=0)=1-p$. Find the following quantities:

a. $E[X]$.

b. $E[X^2]$.

c. $E[5X+3]$

d. $E[4X^2-2]$

e. $E[\sin(\pi X+\pi/2)]$

(a) $E[X] = (1)p + (0)(1-p) = p$

(b) $E[X^2] = (1)^2 p + (0)^2(1-p) = p$

(c) $E[5X+3] = 5E[X]+3 = 5p+3$

(d) $E[4X^2-2] = 4E[X^2]-2 = 4p-2$

(e) $E[\sin(\pi X + \frac{\pi}{2})] = \sin(\pi(1)+\frac{\pi}{2})\cdot p + \sin(\pi(0)+\frac{\pi}{2})\cdot(1-p)$

$= \sin(\frac{3\pi}{2})p + \sin(\frac{\pi}{2})\cdot(1-p)$

$= -p + (1-p) = 1-2p$

Problem 3. Suppose

- $X \sim \text{Unif}\{1,7\}$. This means $P(X=k)=1/2$ for $k=1,7$.
- $Y \sim \text{Unif}\{1,2,3,4,5,6,7\}$. This means $P(Y=k)=1/7$ for $k=1,\ldots,7$.
- $Z=4$. This means Z is a constant; ie. $P(Z=4)=1$.

Find the following quantities

a. $E[X]$

b. $E[Y]$

c. $E[Z]$

d. $E[X^2]$

e. $E[Y^2]$

f. $E[Z^2]$

(a) $E[X] = 1(\frac{1}{2}) + 7(\frac{1}{2}) = 4$

(b) $E[Y] = \frac{1+7}{2} = 4$

(c) $E[Z] = 4$

(d) $E[X^2] = 1(\frac{1}{2}) + 49(\frac{1}{2}) = 25$

(e) $E[Y^2] = (1^2+2^2+3^2+4^2+5^2+6^2+7^2)(\frac{1}{7})$

$= \frac{7(8)(15)}{6}(\frac{1}{7})$

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

$= \frac{4}{3}(15)$

$= 20$

(f) $E[Z^2] = 16$