§ 5.2 Moment Generating Functions

Def The kth moment of a random variable is $E[X^k]$

Remark The moments of a random variable are important for understanding the "shape" of its distribution.

In this section we introduce a new function that gives a way to compute moments (eg. mean, second moment, etc.) of random variables. This will also be an important tool later when we discuss the Central Limit Theorem.

Def Let X be a given random variable. The moment generating function of X is $m_X(t) = E[e^{tX}]$.
It is defined for all $t \in \mathbb{R}$ when the expectation exists.

Getting moments from $m_X(t)$

$$m_X'(t) = \frac{d}{dt} E[e^{tX}]$$

$$= E\left[\frac{d}{dt} e^{tX}\right]$$

$$= E[X e^{tX}]$$

$$\Rightarrow \quad m'(0) = E[X]$$

$$m_X''(t) = \frac{d}{dt} E[X e^{tX}]$$

$$= E\left[\frac{d}{dt} X e^{tX}\right]$$

$$= E[X^2 e^{tX}]$$

$$\Rightarrow \quad m''(0) = E[X^2]$$

In general, $\quad m_X^{(k)}(0) = E[X^k]$ for $k = 1, 2, 3, \ldots$

as long as $E[X^k]$ is finite.

Example Let $X \sim \mathrm{Pois}(\lambda)$. Find the mgf of X and use it to compute $E[X]$ and $V(X)$.

$$\begin{aligned}
m_X(t) &= E[e^{tX}] \\
&= \sum_{k=0}^{\infty} e^{tk} P(X=k) \\
&= \sum_{k=0}^{\infty} e^{tk} e^{-\lambda} \frac{\lambda^k}{k!} \\
&= e^{-\lambda} \sum_{k=0}^{\infty} \frac{(e^t\lambda)^k}{k!} \\
&= e^{-\lambda} e^{e^t\lambda} \qquad \text{since } \sum_{k=0}^{\infty} \frac{x^k}{k!} = e^x \\
&= e^{\lambda(e^t-1)}
\end{aligned}$$

$$\begin{aligned}
m_X'(t) &= e^{\lambda(e^t-1)} \cdot \frac{d}{dt}(\lambda e^t - 1) \qquad \text{(chain rule)} \\
&= e^{\lambda(e^t-1)} \cdot \lambda e^t
\end{aligned}$$

$$\Rightarrow \quad E[X] = m_X'(0) = e^{\lambda(e^0-1)} \lambda e^0 = \lambda$$

$$\begin{aligned}
m_X''(t) &= \frac{d}{dt}\left(e^{\lambda(e^t-1)}\right) \cdot \lambda e^t + e^{\lambda(e^t-1)} \cdot \frac{d}{dt}(\lambda e^t) \qquad \text{(product rule)} \\
&= e^{\lambda(e^t-1)} \cdot \lambda e^t \cdot \lambda e^t + e^{\lambda(e^t-1)} \cdot \lambda e^t
\end{aligned}$$

$$\Rightarrow \quad E[X^2] = m_X''(0) = \lambda^2 + \lambda$$

$$\begin{aligned}
\Rightarrow \quad V(X) &= E[X^2] - E[X]^2 \\
&= \lambda^2 + \lambda - \lambda^2 \\
&= \lambda
\end{aligned}$$

Theorem (properties of mgf)

① If X and Y are independent, then the mgf of $X+Y$ is

$$m_{X+Y}(t) = m_X(t)\, m_Y(t)$$

② If $a, b \in \mathbb{R}$ with $a \neq 0$,

$$m_{aX+b}(t) = e^{bt}\, m_X(at)$$

③ If X and Y have the same mgf, then they have the same distribution.

Example Let $X \sim \mathrm{Pois}(\lambda)$ and $Y \sim \mathrm{Pois}(\mu)$ be independent. Find the distribution of $X+Y$.

Note
$$m_{X+Y}(t) = m_X(t)\, m_Y(t)$$
$$= e^{\lambda(e^t-1)} e^{\mu(e^t-1)}$$
$$= e^{(\lambda+\mu)(e^t-1)}$$

Therefore $X+Y \sim \mathrm{Pois}(\lambda+\mu)$.

Problem 1. Let X be a random variable whose probability mass function is given by

$$P(X=k) = \begin{cases} 1/6 & k=-2 \\ 1/3 & k=0 \\ 1/2 & k=3. \end{cases}$$

a. Find the moment generating function $m(t)$ of X.

b. Compute $E[X]$ and $E[X^2]$ using your answer to part a.

c. Suppose Y is independent of X with the same distribution. Find the moment generating function of $X+Y$ using your answer to part a.

ⓐ $m(t) = E[e^{tX}] = \frac{1}{6}e^{-2t} + \frac{1}{3} + \frac{1}{2}e^{3t}$

ⓑ $m'(t) = \frac{-1}{3}e^{-2t} + \frac{3}{2}e^{3t}$, $\quad E[X] = m'(0) = -\frac{1}{3} + \frac{3}{2} = \frac{7}{6}$

$m''(t) = \frac{2}{3}e^{-2t} + \frac{9}{2}e^{3t}$, $\quad E[X^2] = m''(0) = \frac{2}{3} + \frac{9}{2} = \frac{31}{6}$

ⓒ $E[e^{t(X+Y)}] = E[e^{tX}e^{tY}] = E[e^{tX}]E[e^{tY}] = m(t)^2 = \left(\frac{1}{6}e^{-2t} + \frac{1}{3} + \frac{1}{2}e^{3t}\right)^2$

Problem 2. Let $X \sim \text{Ber}(p)$. Find the moment generating function of X and the first three moments of X.

$$m_X(t) = E[e^{tX}] = e^{t\cdot 1}\cdot p + e^{t\cdot 0}(1-p) = e^t p + 1 - p$$

$$m_X'(t) = e^t p \Rightarrow E[X] = m_X'(0) = p$$

$$m_X''(t) = e^t p \Rightarrow E[X^2] = m_X''(0) = p$$

$$m_X'''(t) = e^t p \Rightarrow E[X^3] = m_X'''(0) = p$$

Problem 3. Let $Y \sim \text{Bin}(n,p)$. Use the moment generating function you found in Problem 2 to find the moment generating function of Y. Remember that $Y = X_1 + X_2 + \cdots + X_n$ where $X_1, X_2, \ldots, X_n \sim \text{Ber}(p)$ are i.i.d.

$$m_Y(t) = m_{X_1}(t)\, m_{X_2}(t) \cdots m_{X_3}(t)$$

$$= (e^t p + 1 - p)^n$$

Problem 4. Let $Y \sim \text{Bin}(n,p)$ and $Z \sim \text{Bin}(m,p)$ be independent random variables. Use the moment generating function you found in Problem 3 to find the moment genrating function of $Y + Z$ and then determine the distribution of $Y + Z$, including any relevant parameters.

$$m_{Y+Z}(t) = m_Y(t)\, m_Z(t)$$

$$= (e^t p + 1 - p)^n (e^t p + 1 - p)^m$$

$$= (e^t p + 1 - p)^{n+m}$$

So $Y + Z \sim \text{Bin}(n+m, p)$.