Problem 1. Let X be a random variable with density given by

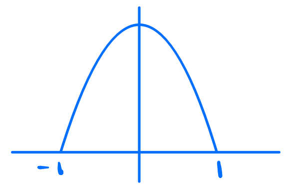
$$f(x) = \begin{cases} c(1-x^2) & -1 < x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

a. Find c

b. Find the CDF of X.

c. Find $P(-.5 \le X < 0.5)$.

d. Find $E[X]$ and $V(X)$.



ⓐ $1 = \int_{-1}^{1} c(1-x^2)\,dx = 2c\int_0^1 (1-x^2)\,dx$

$= 2c\left(x - \frac{1}{3}x^3 \Big|_0^1\right)$

$= \frac{4}{3}c$, so $c = 3/4$

ⓑ when $-1 < x < 1$, $F(x) = P(X \le x) = \int_{-1}^{x} c(1-t^2)\,dt$

$= c\left(t - \frac{1}{3}t^3 \Big|_{-1}^{x}\right)$

$= c\left[\left(x - \frac{1}{3}x^3\right) - \left(-1 + \frac{1}{3}\right)\right]$

$= \frac{3}{4}\left(x - \frac{1}{3}x^3\right) + \frac{1}{2}$.

So $F(x) = \begin{cases} 0 & x \le -1 \\ \frac{3}{4}\left(x - \frac{1}{3}x^3\right) + \frac{1}{2} & -1 < x < 1 \\ 1 & x \ge 1 \end{cases}$

ⓒ $P(-1/2 \le X \le 1/2) = F(1/2) - F(-1/2)$

$= 3/4\,(1/2 - 1/24) - 3/4\,(-1/2 + 1/24)$

$= \frac{11}{16}$

ⓓ $E[X] = \int_{-1}^{1} cx(1-x^2)\,dx = 0$

$V(X) = E[X^2] - E[X]^2 = \int_{-1}^{1} cx^2(1-x^2)\,dx$

$= 2c\int_0^1 (x^2 - x^4)\,dx$

$= 2c\left(\frac{1}{3}x^3 - \frac{1}{5}x^5\right)\Big|_0^1$

$= \frac{4c}{15} = \frac{4}{15}\left(\frac{3}{4}\right) = \frac{1}{5}$

Problem 2. Suppose that X is a random variable with CDF given by

$$F(x) = \begin{cases} 1 - \frac{1}{x^2} & x > 1 \\ 0 & \text{otherwise.} \end{cases}$$

a. Find the median of X.

b. Find $P(X \geq 1.5)$

c. Find $P(3 \leq X \leq 4)$

d. Find the density of X.

(a) find m so that $F(m) = P(X \leq m) = \frac{1}{2}$:

$$1 - \frac{1}{m^2} = \frac{1}{2} \Rightarrow \frac{1}{m^2} = \frac{1}{2} \Rightarrow m^2 = 2 \Rightarrow m = \sqrt{2}$$

(b) $$P(X \geq 1.5) = 1 - P(X < 1.5)$$
$$= 1 - F(1.5)$$
$$= \frac{1}{(1.5)^2} = \frac{4}{9}$$

(c) $$P(3 \leq X \leq 4) = P(X \leq 4) - P(X < 3)$$
$$= F(4) - F(3)$$
$$= \frac{1}{3^2} - \frac{1}{4^2} = \frac{7}{144}$$

(d) $$f(x) = F'(x) = \begin{cases} \frac{2}{x^3} & x > 1 \\ 0 & x \leq 1. \end{cases}$$

Problem 3. Suppose you are working at a blood bank where people donate blood. Type O blood is one of the best to be donated since it can be used for many people. Approximately 42% of people have type O blood.

a. Find the probability that it takes exactly 15 patients until you've gotten 1 with type O blood.

b. Find the probability that it takes exactly 15 patients until you've gotten 3 with type O blood.

c. On average how many patients have to come in until you've gotten 3 with type O blood?

(a) Let $X \sim Geom(0.42)$. $P(X = 15) = (0.58)^{14}(0.42)$

(b) Let $Y \sim NegBin(3, 0.42)$ $P(Y = 15) = \binom{14}{2}(0.58)^{12}(0.42)^3$

(c) $E[Y] = \frac{3}{0.42}$

Problem 4. For the following situations describe the distribution, including parameters, of the given random variables. Give the most reasonable distribution for the situation.

a. Every day there is a 10% chance that Rick will receive no mail. Let X be the number of times he receives mail over the next 5 days.

b. Shawna is playing craps at the casino. The probability of winning craps is about 0.49. She will keep playing until she wins. Let X be the number of times she will play.

c. Rainer is listening to a music station which he isn't used to, and finds that he recognizes about 30% of the songs that they play. He plans to keep listening to songs until he hears 4 songs that he knows. Let X be the number of total songs that Rainer has to listen to in order to hear 4 songs that he knows.

(a) $\text{Bin}(5, 0.9)$

(b) $\text{Geom}(0.49)$

(c) $\text{NegBin}(4, 0.3)$

Problem 5. A hotel has 25 single rooms, all occupied, and numbered 1 to 25. Each room, independent of the others, makes on average 3 phone calls per hour.

a. Give an appropriate distribution for the number of phone calls made by a single room in a one hour period, including any relevant parameters.

b. Give an appropriate distribution for the number of rooms which each made exactly 2 phone calls in a one hour period, including any relevant parameters.

c. Find the probability that

1. at least 1 phone call was made by room 5 in a one hour period.
2. at least 2 phone calls were made in total by rooms 5 and 6 in a one hour period.
3. at least 3 rooms each made exactly 2 phone calls on a one hour period.

(a) $\text{Pois}(3)$

(b) Let $X \sim \text{Pois}(3)$ and let $p = P(X=2) = e^{-3}\frac{3^2}{2!} = 4.5e^{-3}$

Then $Y \sim \text{Bin}(25, p)$ is the appropriate distribution.

(c) (1) $P(X \geq 1) = 1 - P(X=0) = 1 - e^{-3}$

(2) Let $X_1, X_2 \sim \text{Pois}(3)$ be i.i.d. Then

$$P(X_1 + X_2 \geq 2) = 1 - P(X_1 + X_2 \leq 1)$$

$$= 1 - \left(P(X_1=0, X_2=0) + P(X_1=0, X_2=1) + P(X_1=1, X_2=0)\right)$$

$$= 1 - \left((e^{-3})^2 + 2(e^{-3})(e^{-3}\cdot 3)\right)$$

$$= 1 - 7e^{-6}$$

(3) $P(Y \geq 3) = 1 - P(Y \leq 2) = 1 - (1-p)^{25} + 25p(1-p)^{24} + \binom{25}{2}p^2(1-p)^{23}$

Problem 6. Faith is handing out treats to her cat. She has been randomly generating X, the number of treats using a binomial distribution with $n = 3$ and $p = 0.4$. Malcolm says that it's unfair that the cat might get 0 treats, so he suggests 0 be ruled out as an option, and those values all turned into 1's, so that the cat always gets at least one treat. Denote Malcolm's new random variable for the number of treats by U.

a. Find the moment generating function of U.

b. Use the moment generating function of U to compute $E[U]$.

(a)

$$P(U=k) = \begin{cases} (0.6)^3 + 3(0.4)(0.6)^2 & k=1 \\ 3(0.4)^2(0.6) & k=2 \\ (0.4)^3 & k=3 \end{cases} = \begin{cases} 81/125 & k=1 \\ 36/125 & k=2 \\ 8/125 & k=3 \end{cases}$$

$$m(t) = E[e^{tU}] = \frac{81}{125}e^t + \frac{36}{125}e^{2t} + \frac{8}{125}e^{3t}$$

(b)

$$m'(t) = \frac{81}{125}e^t + \frac{72}{125}e^{2t} + \frac{24}{125}e^{3t}$$

$$E[U] = m'(0) = \frac{81}{125} + \frac{72}{125} + \frac{24}{125} = \frac{177}{125}$$

Problem 7. Let X be the value of the first die and Y the sum of the values when two dice are rolled. Compute the moment generating functions of X and Y.

$X \sim \text{Unif}(\{1,2,3,4,5,6\})$

$$m_X(t) = E[e^{tX}] = \frac{1}{6}(e^t + e^{2t} + e^{3t} + e^{4t} + e^{5t} + e^{6t})$$

$Y = X_1 + X_2$ where $X_1, X_2 \sim \text{Unif}(\{1,2,3,4,5,6\})$ are i.i.d.

$$\begin{aligned} m_Y(t) = E[e^{tY}] &= E[e^{t(X_1+X_2)}] \\ &= E[e^{tX_1}]\,E[e^{tX_2}] \\ &= m_X(t)^2 \\ &= \frac{1}{36}(e^t + e^{2t} + e^{3t} + e^{4t} + e^{5t} + e^{6t})^2 \end{aligned}$$

Problem 8. Consider a random variable X whose probability mass function is given below. Find the following quantities.

$$P(X=x) = \begin{cases} c & x=-3 \\ 2c & x=-2 \\ 2c & x=0 \\ 3c & x=1 \\ 4c & x=2. \end{cases}$$

a. c

b. $P(X>-2)$

c. $P(-2 \le X < 2)$

d. $E[X]$

e. $E[X^2]$

(a) $12c = 1 \Rightarrow c = 1/12$

(b) $P(X > -2) = 9c = 3/4$

(c) $P(-2 \le X < 2) = 7c = 7/12$

(d) $E[X] = -3c - 2(2c) + 3c + 2(4c)$

$= 4c = 1/3$

(e) $E[X^2] = 9c + 4(2c) + 3c + 4(4c)$

$= 36c = 3.$

Problem 9. A bag contains 3 red, 5 green, and 7 blue balls. A sample of 2 balls is drawn with replacement. Let X be the number of red balls in the sample and let Y be the number of green balls in the sample.

a. Find the joint probability mass function of X and Y.

b. Find $P(X>Y)$.

c. Find the marginal distributions of X and Y.

d. Are X and Y independent? How do you know?

e. Find $E[X+Y]$.

(a)

	$Y=0$	$Y=1$	$Y=2$	
$X=0$	$(7/15)^2$	$2(5/15)(7/15)$	$(5/15)^2$	$144/225$
$X=1$	$2(3/15)(7/15)$	$2(3/15)(5/15)$	0	$72/225$
$X=2$	$(3/15)^2$	0	0	$9/225$
	$100/225$	$100/225$	$25/225$	

(b) $P(X > Y) = P(X=1, Y=0) + P(X=2, Y=0)$

$$= \frac{42}{225} + \frac{9}{225} = \frac{51}{225}$$

(c) $$P(X=k) = \begin{cases} \frac{144}{225} & k=0 \\ \frac{72}{225} & k=1 \\ \frac{9}{225} & k=2 \end{cases} \qquad P(Y=k) = \begin{cases} \frac{100}{225} & k=0 \\ \frac{100}{225} & k=1 \\ \frac{25}{225} & k=2 \end{cases}$$

(d) $E[X+Y] = E[X] + E[Y]$

$$= \left(\frac{72}{225} + \frac{18}{225}\right) + \left(\frac{100}{225} + \frac{50}{225}\right)$$

$$= \frac{240}{225}$$

Problem 10. Let

$$E[X] = a, E[X^2] = b, E[X^3] = c, E[X^4] = d,$$
$$E[Y] = e, E[Y^2] = f, E[Y^3] = g, E[Y^4] = h.$$

Suppose X and Y are independent. Find the following quantities.

a. $V(X - Y^2)$

b. $E[X^2Y^3]$

c. $V(X^2Y^2)$

(a) $V(X - Y^2) = V(X) + V(Y^2)$

$$= b - a^2 + h - f^2$$

(b) $E[X^2Y^3] = E[X^2]E[Y^3]$

$$= bg$$

(c) $V(X^2Y^2) = E[(X^2Y^2)^2] - E[X^2Y^2]^2$

$$= E[X^4]E[Y^4] - E[X^2]^2 E[Y^2]^2 = dh - b^2f^2$$