## § 8.2 Max and Min of Independent Random Variables

**Example** Suppose reliability of 3 lab computers is being monitored. Let $X_1, X_2, X_3$ be the times until they crash, respectively. We assume these times are independent with parameters $\lambda_1 = 2$, $\lambda_2 = 3$, $\lambda_3 = 5$. Let $M$ be the time of the first one to crash. Find the density (and hence distribution) of $M$.

We'll first find the CDF of $M = \min\{X_1, X_2, X_3\}$.

Observe that

$$
\begin{aligned}
P(M > x) &= P(\min\{X_1, X_2, X_3\} > x) \\
&= P(X_1 > x, X_2 > x, X_3 > x) \\
&= P(X_1 > x)\, P(X_2 > x)\, P(X_3 > x) \\
&= e^{-\lambda_1 x}\, e^{-\lambda_2 x}\, e^{-\lambda_3 x} \\
&= e^{-(\lambda_1 + \lambda_2 + \lambda_3)x}
\end{aligned}
$$

So $F_M(x) = 1 - e^{-(\lambda_1 + \lambda_2 + \lambda_3)x}$.

Therefore $f_M(x) = F_M'(x) = (\lambda_1 + \lambda_2 + \lambda_3)\, e^{-(\lambda_1 + \lambda_2 + \lambda_3)x}$ when $x > 0$ (and $0$ when $x \leq 0$). Therefore $M \sim \mathrm{Exp}(\lambda_1 + \lambda_2 + \lambda_3)$.

Example An insurance company estimates that a payout for an insurance claim is a random variable with density $f(x) = \begin{cases} x^2/9 & 0 < x < 3 \\ 0 & \text{else} \end{cases}$ (in thousands of \$).

Suppose $X_1, \ldots, X_n$ represent payouts from $n$ independent claims. Find the expected value of the maximum payout.

Let $M = \max\{X_1, \ldots, X_n\}$. We'll find the density of $M$ by first considering its CDF.

$$F_M(x) = P(M \le x) = P(\max\{X_1, \ldots, X_n\} \le x)$$

$$= P(X_1 \le x, \ldots, X_n \le x)$$

$$= P(X_1 \le x) \cdots P(X_n \le x)$$

$$= F_{X_1}(x) \cdots F_{X_n}(x)$$

$$= (F_{X_1}(x))^n$$

Therefore

$$f_M(x) = F_M'(x) = n\,(F_{X_1}(x))^{n-1} \cdot F_{X_1}'(x)$$

$$= n\,(F_{X_1}(x))^{n-1} \cdot f_{X_1}(x)$$

$$= n\,(F_{X_1}(x))^{n-1} \cdot \frac{x^2}{9}$$

Note $F_{X_1}(x) = \int_0^x \frac{t^2}{9}\,dt = \frac{1}{27}x^3$. Thus

$$f_M(x) = n\left(\frac{1}{27}x^3\right)^{n-1} \cdot \frac{x^2}{9}$$

$$= \frac{n\,x^{3n-3} \cdot x^2}{27^{n-1} \cdot 9} = \frac{3n x^{3n-1}}{27^n}$$

$$E[M] = \int_0^3 x \cdot \frac{3n\,x^{3n-1}}{27^n}\,dx$$

$$= \frac{3n}{27^n} \cdot \int_0^3 x^{3n}\,dx = \frac{3n}{27^n} \cdot \frac{3^{3n+1}}{3n+1} = \frac{9n}{3n+1}$$

Curious observation Since $\lim_{n\to\infty} \frac{9n}{3n+1} = 3$, as number of claims increases, expected max payout approaches \$3000.

**Problem 1.** Suppose we pick 4 random numbers in the interval $(0,1)$ independently. Let $M_1$ be their minimum and let $M_2$ be their maximum.

a. Find the CDFs of $M_1$ and $M_2$.

b. Find the densities of $M_1$ and $M_2$.

c. Find the probability that the smallest number is greater than $1/4$.

d. Find the probability that the biggest number is less than $1/2$.

e. Find the expected value of the smallest number.

f. Find the expected value of the biggest number.

(a) $M_1 = \min\{X_1, X_2, X_3, X_4\}$, $M_2 = \max\{X_1, X_2, X_3, X_4\}$

where $X_1, \dots, X_4 \sim \text{Unif}(0,1)$ are i.i.d.

$$F_{M_1}(x) = P(M_1 \le x) = 1 - P(M_1 > x)$$

$$= 1 - P(X_1 > x, X_2 > x, X_3 > x, X_4 > x)$$

$$= 1 - P(X_1 > x)^4$$

$$= 1 - (1-x)^4 \quad \text{when } 0 < x < 1$$

$$F_{M_2}(x) = P(M_2 \le x) = P(X_1 \le x, X_2 \le x, X_3 \le x, X_4 \le x)$$

$$= P(X_1 \le x)^4$$

$$= x^4 \quad \text{when } 0 < x < 1.$$

ⓑ $f_{M_1}(x) = F_{M_1}'(x) = \begin{cases} 4(1-x)^3 & 0<x<1 \\ 0 & \text{else} \end{cases}$

$f_{M_2}(x) = F_{M_2}'(x) = \begin{cases} 4x^3 & 0<x<1 \\ 0 & \text{else.} \end{cases}$

ⓒ $P(M_1 > 1/4) = (1-1/4)^4 = (3/4)^4 = \frac{81}{256}$

ⓓ $P(M_2 < 1/2) = (1/2)^4 = \frac{1}{16}$

ⓔ $E[M_1] = \int_0^1 4x(1-x)^3\,dx$ $\quad u = 1-x$, $du = -dx$

$= \int_0^1 4(1-u)u^3\,du$

$= \int_0^1 (4u^3 - 4u^4)\,du$

$= u^4 - \frac{4}{5}u^5 \Big|_0^1$

$= 1 - \frac{4}{5} = \frac{1}{5}$

ⓕ $E[M_2] = \int_0^1 4x^4\,dx = \frac{4}{5}x^5\Big|_0^1 = \frac{4}{5}$

**Problem 2.** Suppose the lifetime for a certain brand of lightbulb is modeled with the exponential distribution with a mean lifetime of 5 years. Suppose further that we buy 10 such bulbs and assume that their lifetimes are independent. Let $M$ be the time until the first one dies.

a. Find $P(M > 5)$.

b. Find $E[M]$.

$M = \min\{X_1, \dots, X_{10}\} \sim Exp(1/5 + \cdots + 1/5) = Exp(2)$

when $X_1, \dots, X_{10} \sim Exp(1/5)$ are i.i.d.

ⓐ $P(M>5) = 1 - P(M \le 5) = 1 - (1 - e^{-5\lambda}) = e^{-10}$

ⓑ $E[M] = \frac{1}{2}$.