§10.6 Proof of the CLT

Today we'll work on proving the Central Limit Theorem.

There are two technical results we'll discuss first.

Def A limit is of indeterminate form if it has one of following informal expressions: $\frac{0}{0}$, $\frac{\infty}{\infty}$, $0 \cdot \infty$, 0^0, 1^∞, ∞^0, $\infty - \infty$

Theorem (L'Hôpital's Rule) If f and g are differentiable functions such that $\lim\limits_{x \to a} f(x) = \lim\limits_{x \to a} g(x) = 0$ or

$\lim\limits_{x \to a} f(x) = \pm \infty = \lim\limits_{x \to a} g(x)$ and $\lim\limits_{x \to a} \frac{f'(x)}{g'(x)} = L$, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = L.$$

Example Compute ① $\lim\limits_{x \to 1} \frac{\ln x}{x-1}$ ② $\lim\limits_{x \to \infty} \frac{e^x}{x^2}$ ③ $\lim\limits_{x \to \infty} x \sin\left(\frac{1}{x}\right)$.

① $\lim\limits_{x \to 1} \frac{\ln x}{x-1} \overset{L'H}{=} \lim\limits_{x \to 1} \frac{1/x}{1} = 1$

② $\lim\limits_{x \to \infty} \frac{x^2}{e^x} \overset{L'H}{=} \lim\limits_{x \to \infty} \frac{2x}{e^x} \overset{L'H}{=} \lim\limits_{x \to \infty} \frac{2}{e^x} =$

③ $\lim\limits_{x \to \infty} x \cdot \sin\left(\frac{1}{x}\right) = \lim\limits_{x \to \infty} \frac{\sin\left(\frac{1}{x}\right)}{\frac{1}{x}} \overset{L'H}{=} \lim\limits_{x \to \infty} \frac{\cos\left(\frac{1}{x}\right) \cdot \frac{-1}{x^2}}{\frac{-1}{x^2}} = 1$

Theorem (Lévy Continuity Theorem)

Suppose $\bar{Y}_1, \bar{Y}_2, \ldots$ is a sequence of random variables with moment generating functions $m_1(t), m_2(t), \ldots$ and let $\bar{Y}$ be a random variable with mgf $m_{\bar{Y}}(t)$.

If $\lim_{n\to\infty} m_n(t) = m_{\bar{Y}}(t)$ for all t, then the sequence $(\bar{Y}_n)_{n\geq 1}$ converges in distribution to $\bar{Y}$. That is

$\lim_{n\to\infty} P(\bar{Y}_n \leq x) = P(\bar{Y} \leq x)$ (the CDF's of $\bar{Y}_n$ converge to the CDF of $\bar{Y}$). In other words, to prove convergence in distribution it suffices to prove convergence of mgf's.

Intuition The mgf of a random variable characterize all of its moment which in turn characterize the (shape of the) distribution and its CDF.

Corollary (CLT) Let $X_1, X_2, \ldots$ be an i.i.d. sequence of random variables with finite mean $\mu = E[X_i]$ and variance $\sigma^2 = E[X_i]$ and let $S_n = X_1 + \cdots + X_n$ for each $n \geq 1$. Then $\dfrac{\frac{S_n}{n} - \mu}{\sigma/\sqrt{n}}$ converges in distribution to $Z \sim N(0,1)$. That is,

$$\lim_{n \to \infty} P\left(\frac{\frac{S_n}{n} - \mu}{\sigma/\sqrt{n}} \leq x \right) = P(Z \leq x) \quad \text{for all } x \in \mathbb{R}.$$

Sketch of proof

Let $m_n(t)$ be the mgf of $\dfrac{\frac{S_n}{n} - \mu}{\sigma/\sqrt{n}}$ for each $n \geq 1$ and prove that $m_n(t)$ converges to the mgf of $Z \sim N(0,1)$. Do this in two cases:

1. the case when $\mu = 0$, $\sigma^2 = 1$
2. the case of general μ and σ^2.

A few important reminders about mgf's

1. $m_Z(t) = e^{t^2/2}$ when $Z \sim N(0,1)$
2. $m_X^{(k)}(0) = E[X^k]$
3. $m_{X+Y}(t) = m_X(t)^2$ when X, Y are i.i.d.
4. $m_{aX}(t) = m_X(at)$ for any constant $a \neq 0$.

① $\dfrac{\frac{S_n}{n} - \mu}{\sigma/\sqrt{n}} = \dfrac{\frac{S_n}{n} - 0}{1/\sqrt{n}} = \dfrac{S_n}{\sqrt{n}}$

② $m_n(t) = E\left[e^{t\left(\frac{S_n}{\sqrt{n}}\right)} \right]$

$= E\left[e^{\frac{t}{\sqrt{n}}(X_1 + \cdots + X_n)} \right]$

$= E\left[e^{\frac{t}{\sqrt{n}} X_1} \cdots e^{\frac{t}{\sqrt{n}} X_n} \right]$

$= m_{X_1}\left(\frac{t}{\sqrt{n}}\right) \cdots m_{X_n}\left(\frac{t}{\sqrt{n}}\right)$

$= m\left(\frac{t}{\sqrt{n}}\right)^n$

③ $m(0) = E[e^{0 X_i}] = 1$

$m'(0) = E[X_i] = \mu = 0$

$m''(0) = E[X_i^2] = V(X_i) + E[X_i]^2 = \sigma^2 + \mu^2 = 1$

④ $\underline{1}^{\infty}$

⑤ $$\lim_{n\to\infty} \ln(m_n(t)) = \lim_{n\to\infty} \ln\left(m\left(\tfrac{t}{\sqrt{n}}\right)^n \right)$$

$$= \lim_{n\to\infty} n \ln\left(m\left(\tfrac{t}{\sqrt{n}}\right)\right)$$

$$= \lim_{n\to\infty} \frac{\ln\left(m\left(\frac{t}{\sqrt{n}}\right)\right)}{\frac{1}{n}}$$

$$\overset{L'H}{=} \lim_{n\to\infty} \frac{\frac{m'\left(\frac{t}{\sqrt{n}}\right)}{m\left(\frac{t}{\sqrt{n}}\right)} \cdot \left(-\frac{1}{2}\right) \frac{t}{n^{3/2}}}{-\frac{1}{n^2}}$$

$$= \frac{t}{2} \lim_{n\to\infty} \frac{m'\left(\frac{t}{\sqrt{n}}\right)}{m\left(\frac{t}{\sqrt{n}}\right)} \cdot \frac{n^2}{n^{3/2}}$$

$$= \frac{t}{2} \lim_{n\to\infty} m'\left(\tfrac{t}{\sqrt{n}}\right) \cdot n^{1/2}$$

$$= \frac{t}{2} \lim_{n\to\infty} \frac{m'\left(\frac{t}{\sqrt{n}}\right)}{n^{-1/2}}$$

$$\overset{L'H}{=} \frac{t}{2} \lim_{n\to\infty} \frac{m''\left(\frac{t}{\sqrt{n}}\right) \cdot \left(-\frac{1}{2}\right)\frac{t}{n^{3/2}}}{-\frac{1}{2} n^{-3/2}}$$

$$= \frac{t^2}{2} \lim_{n\to\infty} m''\left(\tfrac{t}{\sqrt{n}}\right)$$

$$= \frac{t^2}{2} m''(0) = \frac{t^2}{2}.$$

Therefore, $\lim_{n\to\infty} m_n(t) = e^{t^2/2} = m_Z(t)$ and so

$\frac{S_n}{\sqrt{n}}$ converges in distribution to $Z \sim N(0,1)$ by

the Lévy continuity theorem.

(6) $E[X_i^*] = E\left[\frac{X_i-\mu}{\sigma}\right] = \frac{1}{\sigma}(E[X_i]-\mu) = \frac{1}{\sigma}(\mu-\mu) = 0.$

$V(X_i^*) = V\left(\frac{X_i-\mu}{\sigma}\right) = \frac{1}{\sigma^2}V(X_i-\mu) = \frac{1}{\sigma^2}V(X_i) = 1$

$\lim_{n\to\infty} m_n^*(t) = e^{t^2/2}$ by Problem 5.

(7)
$$\frac{S_n^*}{\sqrt{n}} = \frac{X_1^* + \cdots + X_n^*}{\sqrt{n}}$$

$$= \frac{\left(\frac{X_1-\mu}{\sigma}\right) + \left(\frac{X_2-\mu}{\sigma}\right) + \cdots + \left(\frac{X_n-\mu}{\sigma}\right)}{\sqrt{n}}$$

$$= \frac{\frac{1}{\sigma}(X_1 + \cdots + X_n - n\mu)}{\sqrt{n}}$$

$$= \frac{S_n - n\mu}{\sigma\sqrt{n}} \cdot \frac{\frac{1}{n}}{\frac{1}{n}}$$

$$= \frac{\frac{S_n}{n} - \mu}{\sigma/\sqrt{n}}$$

Therefore $m_n(t) = m_n^*(t)$ and so

$\lim_{n\to\infty} m_n(t) = \lim_{n\to\infty} m_n^*(t) = e^{t^2/2}$, which implies

$\frac{\frac{S_n}{n}-\mu}{\sigma/\sqrt{n}}$ converges in distribution to $Z \sim N(0,1)$.