Problem 1. Let X and Y have joint density given by

$$f(x,y) = \begin{cases} cx & 0 < y < x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

a. Find c.

b. Set up a double integral for $P(X + Y \leq 1)$.

c. Find the marginal density of X.

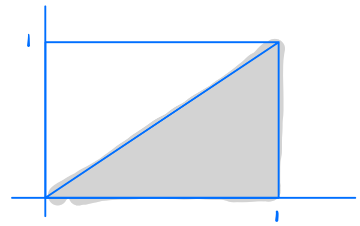
d. Find the marginal density of Y.

e. Are X and Y independent?

f. Find $P(X < 0.75 \mid Y = 0.25)$.

g. Find $P(Y > 0.2 \mid X = 0.75)$.

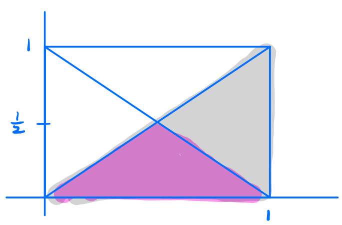
h. Find $E[Y \mid X]$



(a) $$1 = \int_{x=0}^{x=1} \int_{y=0}^{y=x} cx\,dy\,dx$$

$$= \int_0^1 cx^2\,dx = \frac{c}{3} \Rightarrow c = 3.$$

(b)



$$\int_{y=0}^{y=\frac{1}{2}} \int_{x=y}^{x=1-y} 3x\,dx\,dy$$

(c) when $0<x<1$, $f_X(x) = \int_{y=0}^{y=x} 3x\,dy = 3x^2$

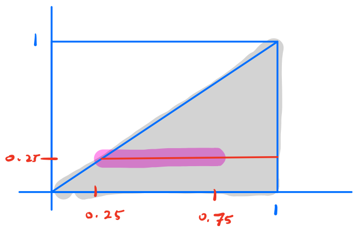
$$f_X(x) = \begin{cases} 3x^2 & 0<x<1 \\ 0 & \text{otherwise} \end{cases}$$

(d) when $0<y<1$, $f_Y(y) = \int_{x=y}^{x=1} 3x\,dx = \frac{3}{2}x^2\Big|_y^1 = \frac{3}{2}(1-y^2)$

$$f_Y(y) = \begin{cases} \frac{3}{2}(1-y^2) & 0<y<1 \\ 0 & \text{otherwise.} \end{cases}$$

(e) no, since $f(x,y) \neq f_X(x) f_Y(y)$

(f) $P(X < 0.75 \mid Y = 0.25) = \int_{x=0.25}^{0.75} f_{X|Y}(x \mid 0.25)\, dx$

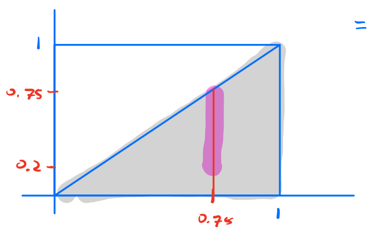


$$= \int_{0.25}^{0.75} \frac{f(x, 0.25)}{f_Y(0.25)}\, dx$$

$$= \frac{\int_{0.25}^{0.75} 3x\, dx}{\frac{3}{2}(1 - 0.25^2)} = \frac{\frac{3}{2}(0.75^2 - 0.25^2)}{\frac{3}{2}(1 - 0.25^2)}$$

$$= \frac{8}{15}$$

(g) $P(Y > 0.2 \mid X = 0.75) = \int_{0.2}^{0.75} f_{Y|X}(y \mid 0.75)\, dy$



$$= \int_{0.2}^{0.75} \frac{f(y, 0.75)}{f_X(0.75)}\, dy$$

$$= \frac{\int_{0.2}^{0.75} 3(0.75)\, dy}{3(0.75)^2}$$

$$= \frac{3(0.75)(0.55)}{3(0.75)^2} = \frac{55}{75} = \frac{11}{15}$$

(h) $E[Y \mid X = x] = \int_{y=0}^{y=x} y\, f_{Y|X}(y \mid x)\, d = \int_0^x y \cdot \frac{f(x,y)}{f_X(x)}\, dy$

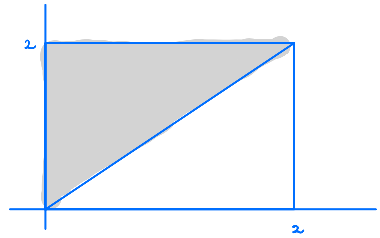
$$= \int_0^x y \cdot \frac{3x}{3x^2}\, dy = \frac{1}{x} \int_0^x y\, dx$$

$$= \frac{1}{2x}\, y^2 \Big|_0^x = \frac{x}{2}.$$

So $E[Y \mid X] = \frac{X}{2}$.

Problem 2. Repeat the previous problem with the joint density

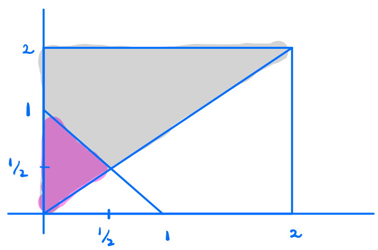
$$f(x,y) = \begin{cases} cxy & 0 < x < y < 2 \\ 0 & \text{otherwise.} \end{cases}$$



(a) $1 = \int_{x=0}^{x=2} \int_{y=x}^{y=2} cxy\,dy\,dx = \int_0^2 \frac{cx}{2}(4-x^2)\,dx$

$= \frac{c}{2}\int_0^2 (4x - x^3)\,dx = \frac{c}{2}\left(2x^2 - \frac{1}{4}x^4 \Big|_0^2\right) = \frac{c}{2}(8-4) = 2c \Rightarrow c = \frac{1}{2}$

(b)



$\int_{x=0}^{x=\frac{1}{2}} \int_{y=x}^{y=1-x} \frac{1}{2}xy\,dy\,dx$

(c) when $0<x<2$, $f_X(x) = \int_{y=x}^{y=2} \frac{1}{2}xy\,dy = \frac{1}{4}x\left(y^2\Big|_x^2\right) = \frac{1}{4}x(4-x^2)$

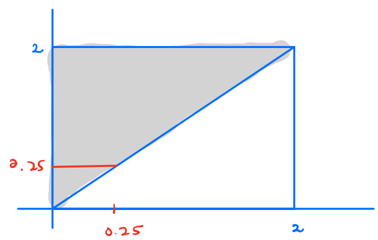
$$f_X(x) = \begin{cases} x - \frac{1}{4}x^3 & 0<x<2 \\ 0 & \text{otherwise} \end{cases}$$

(d) when $0<y<2$, $f_Y(y) = \int_{x=0}^{x=y} \frac{1}{2}xy\,dx = \frac{1}{4}y\left(x^2\Big|_0^y\right) = \frac{1}{4}y^3$

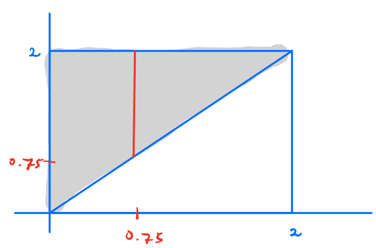
$$f_Y(y) = \begin{cases} \frac{1}{4}y^3 & 0<y<2 \\ 0 & \text{otherwise.} \end{cases}$$

(e) no, since $f(x,y) \neq f_X(x) f_Y(y)$

(f) $P(X < 0.75 \mid Y = 0.25) = 1.$



(g) $P(Y > 0.2 \mid X = 0.75) = 1.$



(h) $E[Y \mid X = x] = \int_{y=x}^{y=2} y \, f_{Y|X}(y|x) \, dy = \int_x^2 y \, \frac{f(x,y)}{f_X(x)} \, dy$

$$= \int_x^2 y \cdot \frac{\frac{1}{2}xy}{\frac{1}{4}x(4-x^2)} \, dy$$

$$= \frac{2}{4-x^2} \int_x^2 y^2 \, dy$$

$$= \frac{2}{4-x^2} \cdot \frac{1}{3} y^3 \Big|_x^2$$

$$= \frac{2}{3(4-x^2)} (8 - x^3)$$

So $E[Y \mid X] = \dfrac{2(8-X^3)}{3(4-X^2)}$

Problem 3. Suppose that X is uniform on the interval $(0,3)$ and that $Y \mid X = x$ is uniform on the interval $(0, x^2)$. Find the following

a. $P(Y < 4 \mid X = x)$ when $0 < x < 3$

b. $P(Y < 4)$

c. $E[Y \mid X = x]$ when $0 < x < 3$

d. $E[Y]$

(a) $$P(Y < 4 \mid X = x) = \begin{cases} 1 & 0 < x < 2 \\ \frac{4}{x^2} & 2 < x < 3 \end{cases}$$

(b) $$P(Y<4) = \int_0^3 P(Y<4 \mid X=x) f_X(x)\,dx$$

$$= \frac{1}{3}\int_0^3 P(Y<4 \mid X=x)\,dx$$

$$= \frac{1}{3}\left(\int_0^2 P(Y<4 \mid X=x)\,dx + \int_2^3 P(Y<4 \mid X=x)\,dx\right)$$

$$= \frac{1}{3}\left(\int_0^2 1\,dx + \int_2^3 \frac{4}{x^2}\,dx\right)$$

$$= \frac{1}{3}\left(2 + 4x^{-1}\Big|_3^2\right) = \frac{2}{3} + \frac{4}{3}\left(\frac{1}{2} - \frac{1}{3}\right)$$

$$= \frac{2}{3} + \frac{4}{3}\left(\frac{1}{6}\right) = \frac{8}{9}.$$

(c) $$E[Y \mid X = x] = \frac{1}{2}x^2$$

(d) $$E[Y] = E\left[\frac{1}{2}X^2\right] = \frac{1}{2}\int_0^3 x^2 \cdot f_X(x)\,dx$$

$$= \frac{1}{6}\int_0^3 x^2\,dx = \frac{1}{18}x^3\Big|_0^3 = \frac{3}{2}.$$

Problem 4. Consider a random variable X with probability mass function given by

$$P(X = x) = \begin{cases} 1/5 & x = 10 \\ 3/10 & x = 20 \\ 1/2 & x = 30. \end{cases}$$

Given $X = x$, we flip a fair coin x times and let Y count the number of heads that result.

a. Give the conditional distribution of Y given $X = x$. Make sure to specify parameters.

b. Find $E[Y|X]$.

c. Find $E[Y]$.

(a) $Y \mid X = x \sim Bin(x, \frac{1}{2})$

(b) $E[Y \mid X] = \frac{1}{2} X$

(c) $E[Y] = E[\frac{1}{2} X] = \frac{1}{2} E[X]$

$= \frac{1}{2} \left(10 \cdot \frac{1}{5} + 20 \cdot \frac{3}{10} + 30 \cdot \frac{1}{2} \right)$

$= \frac{1}{2} (2 + 6 + 15) = \frac{23}{2}$.

Problem 5. Someone receives on average 5 text messages per hour. Let X denote the time until they receive a message.

a. What distribution should be used to model X? What parameter(s) does it have?

b. Find the expected time until receiving a message.

c. Find the probability that it takes between 10 and 20 minutes before receiving a message.

d. Find the probability that it takes less 30 minutes to receive a message.

e. Suppose no message has been received after 30 minutes. Find the probability that still no message has been received after 5 more minutes.

(a) $X \sim Exp(5)$ (in hours) or $X \sim Exp(1/12)$ (in minutes)

(b) $1/5$ hour or 12 minutes

(c) $P(10 \le X \le 20) = F(20) - F(10)$

$= (1 - e^{-20\lambda}) - (1 - e^{-10\lambda}) = e^{-10/12} - e^{-20/12}$

(d) $P(X \le 30) = F(30) = 1 - e^{-30\lambda} = 1 - e^{-30/12}$

(e) $P(X > 35 \mid X > 30) = P(X > 5) = 1 - P(X \le 5) = 1 - F(5) = e^{-5/12}$

Problem 6. Suppose that X is a normal random variable with mean 10 and variance σ^2. Suppose $P(X > 15) = 0.16$. Use the 68-95-99.7 rule to estimate the following quantities.

a. σ^2

b. $P(X < 5)$

c. $P(0 < X < 20)$

d. $P(X > 25)$

(a) $0.16 = P(X > 15) = P\left(\frac{X-10}{\sigma} > \frac{15-10}{\sigma}\right) = P\left(Z > \frac{5}{\sigma}\right)$

Since $P(|Z| \le 1) = 0.68$, $P(Z > 1) = 0.16$, so $\frac{5}{\sigma} = 1$ and $\sigma = 5$

(b) $P(X < 5) = P\left(\frac{X-10}{\sigma} < \frac{5-10}{\sigma}\right) = P(Z < -1) = 0.16$

(c) $P(0 < X < 20) = P\left(\frac{0-10}{\sigma} < Z < \frac{20-10}{\sigma}\right) = P(|Z| \le 2) = 0.95$

(d) $P(X > 25) = P\left(Z > \frac{25-10}{\sigma}\right) = P(Z > 3) = \frac{1-0.997}{2}$

Problem 7. Suppose X and Y are i.i.d. normally distributed random variables random variables with mean μ and variance σ^2. Give the distribution, including any parameters, of the following random variables.

a. $4X + 7$

b. $-3Y + 5$

c. $X + Y$

d. $X - Y$

(a) $N(4\mu + 7, 16\sigma^2)$

(b) $N(-3\mu + 5, 9\sigma^2)$

(c) $N(2\mu, 2\sigma^2)$

(d) $N(0, 2\sigma^2)$

Problem 8. Explain how to use Monte Carlo simulation to approximate the following integrals.

a. $\int_0^1 \sin(x^{-x})\,dx$

b. $\int_1^{10} \sin(x^{-x})\,dx$

c. $\int_0^{\infty} \sin(x^{-x})\,dx$

(a) ① Sample $X_1, \dots, X_n \sim Unif(0,1)$

② Compute $\dfrac{\sin(X_1^{-X_1}) + \cdots + \sin(X_n^{-X_n})}{n}$

(b) ① Sample $X_1, \dots, X_n \sim Unif(1,10)$

② Compute $\dfrac{9\sin(X_1^{-X_1}) + \cdots + 9\sin(X_n^{-X_n})}{n}$

(c) ① Sample $X_1, \dots, X_n \sim Exp(1)$

② Compute $\dfrac{e^{X_1}\sin(X_1^{-X_1}) + \cdots + e^{X_n}\sin(X_n^{-X_n})}{n}$

Problem 9. Use the Central Limit Theorem to give the approximate distribution of the random variable X in the following scenarios.

a. The local farm packs its tomatoes in crates. Individual tomatoes have mean weight of 10 ounces and standard deviation 3 ounces. Let X be the weight of a crate of 50 tomatoes.

b. For one \$1 red bet, let G be the casino's gain. Then $P(G = 1) = 20/38$ and $P(G = -1) = 18/38$. Suppose in 1 day, 1000 red bets are placed. Let X be the casino's total gain.

c. Let X be the average of 100 random numbers sampled from the distribution with density $f(x) = 6x(1-x)$ supported on the interval $(0, 1)$.

(a) $\mu = 10,\ \sigma^2 = 9,\ n = 50,\quad N(n\mu, n\sigma^2)$

(b) $\mu = \frac{2}{38} = \frac{1}{19} \qquad \sigma^2 = 1 - \frac{1}{19^2} = \frac{360}{31},\quad n = 1000 \qquad N(n\mu, n\sigma^2)$

(c) $\mu = \int_0^1 x f(x)\,dx = \frac{1}{2},$

$E[X^2] = \int_0^1 x^2 f(x)\,dx \qquad n = 100,\quad N\left(\mu, \frac{\sigma^2}{n}\right)$

$= \int_0^1 6x^3(1-x)\,dx$

$= \frac{3}{2} - \frac{6}{5} = \frac{3}{10},\qquad \sigma^2 = \frac{3}{10} - \left(\frac{1}{2}\right)^2 = \frac{1}{20}$