Math 342 — Proof of the CLT

Let $X_1, X_2, \ldots$ be an i.i.d. sequence of random variables with mean $\mu = E[X_i]$ and variance $\sigma^2 = V(X_i)$. For each $n \geq 1$, let $S_n = X_1 + \cdots + X_n$ and let $m_n(t)$ be moment generating function of

$$\frac{S_n/n - \mu}{\sigma/\sqrt{n}}.$$

Our goal is to prove that $m_n(t)$ converges to the moment generating function $m_Z(t) = e^{t^2/2}$ of $Z \sim N(0,1)$ as $n \to \infty$. We will break this into two cases. In Problems 1 to 5, we will work with the case where $\mu = 0$ and $\sigma^2 = 1$. Problems 6 and 7 will be for the general case of μ and σ^2.

Problem 1. Give a simplified expression for the random variable $\frac{S_n/n-\mu}{\sigma/\sqrt{n}}$ when $\mu = 0, \sigma^2 = 1$.

Problem 2. Let $m(t) = E[e^{tX_i}]$ denote the moment generating function of each X_i. Show that

$$m_n(t) = m\left(\frac{t}{\sqrt{n}}\right)^n.$$

Problem 3. Let $m(t) = E[e^{tX_i}]$. Give the values of $m(0), m'(0), m''(0)$. *Note that no calculation is necessary.*

Problem 4. If you try to compute

$$\lim_{n\to\infty} m_n(t) = \lim_{n\to\infty} m\left(\frac{t}{\sqrt{n}}\right)^n$$

directly you get an indeterminate form. State the indeterminate form and explain why L'Hôpital's rule cannot be applied directly to compute this limit.

Problem 5. Our goal now is to show that

$$\lim_{n\to\infty} m_n(t) = e^{t^2/2}.$$

We will start by doing an algebraic manipulation, noting that $m_n(t) = e^{\ln m_n(t)}$, and instead we will compute $\lim_{n\to\infty} \ln m_n(t)$. The first steps are shown below. Complete the rest of the calculation and state a conclusion summarizing why the Central Limit Theorem has now been proved in the special case when $\mu = 0$ and $\sigma^2 = 1$.

$$\begin{aligned}
\lim_{n\to\infty} \ln m_n(t) &= \lim_{n\to\infty} \ln m\left(\frac{t}{\sqrt{n}}\right)^n \\
&= \lim_{n\to\infty} n \ln m\left(\frac{t}{\sqrt{n}}\right) = \ldots
\end{aligned}$$

Problem 6. Suppose the mean and variance μ and σ^2 of X_i are not necessarily 0 and 1 respectively. Define

$$X_i^* = \frac{X_i - \mu}{\sigma} \quad \text{and} \quad S_n^* = X_1^* + \cdots + X_n^*$$

and let $m_n^*(t)$ be the moment generating function of $S_n^*/\sqrt{n}$.

a. Find the mean and variance of X_i^*.

b. Find $\lim_{n\to\infty} m_n^*(t)$.

Problem 7. Prove that

$$\frac{S_n^*}{\sqrt{n}} = \frac{S_n/n - \mu}{\sigma/\sqrt{n}}$$

and use this fact to find $\lim_{n\to\infty} m_n(t)$. State a conclusion summarizing why the Central Limit Theorem has now been proved.