A Flow-Kick FitzHugh-Nagumo Model with Stochastic Perturbation

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FitzHugh-Nagumo model

- The FitzHugh-Nagumo model is a two-dimensional differential equations model derived from the Hodgkin-Huxley model for neuron spikes in squid giant axons. The system consists of two coupled differential equations, shown below, where the x-variable represents a membrane potential (voltage) in an excitable cell and the y-variable is an abstract model for the ion channel in the cell that causes the x-variable to settle after spiking.

$$\dot{x} = -x(x-1)(x-2) - y$$
$$\dot{y} = 0.1(x-0.2) + r$$

- When $r = 0$, the system has a stable equilibrium at the intersection of $x = 0.2$ and $y = -x(x-1)(x-2)$. However, different initial conditions can result in qualitatively different trajectories of the solution. See Figure 1, which shows trajectories with initial conditions $(0.15, -0.5)$ and $(0.15, -0.45)$.
- As r approaches $r_0 = -0.08 + 0.1\sqrt{3}^{-1}$ (this is the value such that the vertical nullcline intersects the cubic nullcline at its local minimum), the system undergoes what is known as a Hopf bifurcation. For $r > r_0$, the system has a stable equilibrium point that is converged to by solutions that start nearby. For $r < r_0$, the equilibrium point is no longer stable. Instead, solutions converge to a stable **limit cycle**, which is a closed trajectory representing steady periodic behavior. See Figure 3.

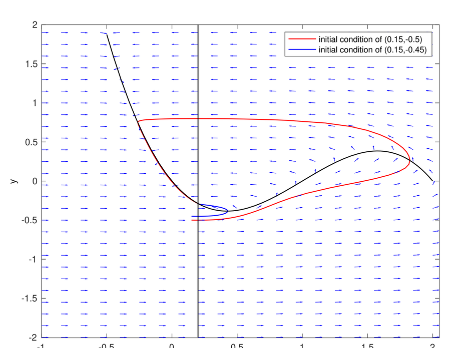


Figure 1. Phase portraits

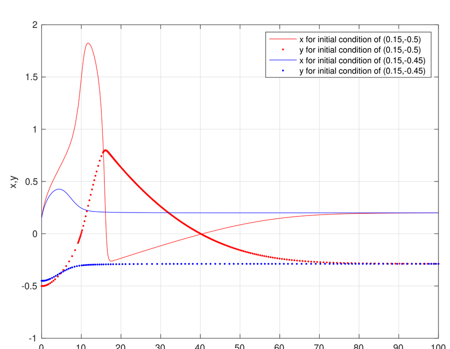


Figure 2. Time series

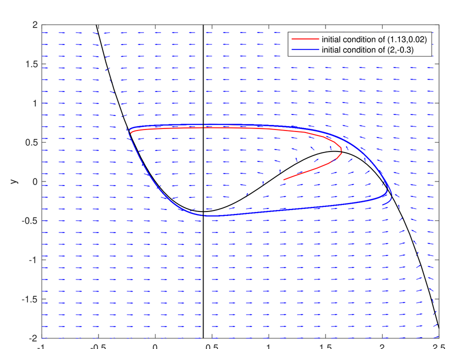


Figure 3. Hopf bifurcation

Stochastic flow-kick FitzHugh-Nagumo model

- Our aim is to study what we call a stochastic flow-kick version of the FHN model defined above. With $r = 0$ and some given initial condition, we let the system evolve for a random time τ called the **flow time**. Then, we instantaneously perturb the y-variable of the solution by a random amount κ called the **kick**. The system then continues to evolve and this process of flow then kick is repeated again and again. The flow times are exponentially distributed, the kicks are uniformly distributed, and all are independent.
- Consider the system with fixed kick distribution $\mathrm{Unif}(-0.1, 0)$ and flow time distribution $\mathrm{Exp}(\bar{\tau}^{-1})$, where $\bar{\tau}$ represents the mean flow time. Qualitatively different behaviors can occur for different values of $\bar{\tau}$.
- If $\bar{\tau}$ is small (a notion to be explored in the rest of the poster), frequent kicks result in many spikes of the x-variable, which we refer to as **excursions**. In contrast, if $\bar{\tau}$ is large, excursions are less likely to occur. More precisely, an *excursion* is defined to occur when x surpasses the threshold $r_0 = 1 + \sqrt{3}^{-1}$ (the maximum of the x-nullcline). See Figure 4 for phase portraits and time series plots exhibiting some possible behaviors.

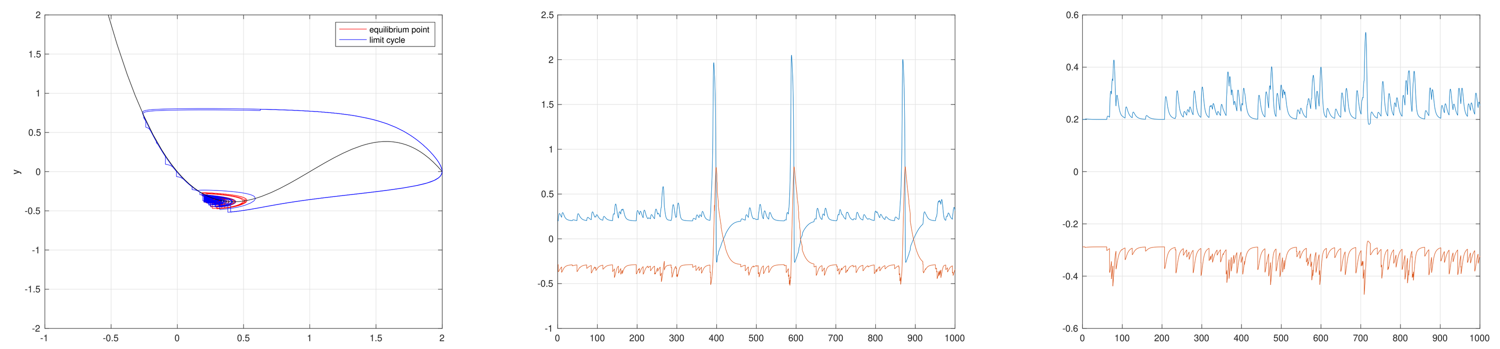


Figure 4. The left pane shows phase portraits corresponding to excursion and no excursion. The middle and right panes show time series corresponding to blue curve and red curve respectively.

Objectives

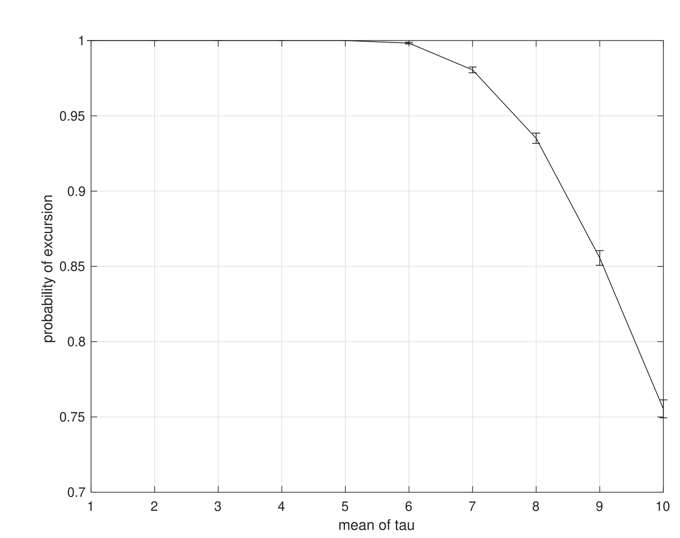
Investigate the effect of varying $\bar{\tau}$ on:

- probability of excursion
- distribution of the interspike interval time σ
- distribution of the excursion time χ spent orbiting a limit cycle

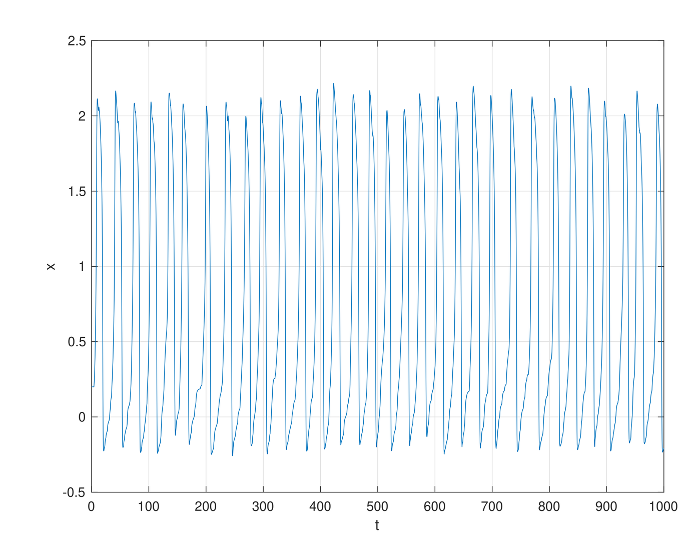
Results

As $\bar{\tau}$ increases, the probability of an excursion happening within a fixed time interval decreases. Figure 5(a) shows this decreasing trend when the system is run for 1000 time units and $\bar{\tau} = 1, 2, \ldots, 10$.

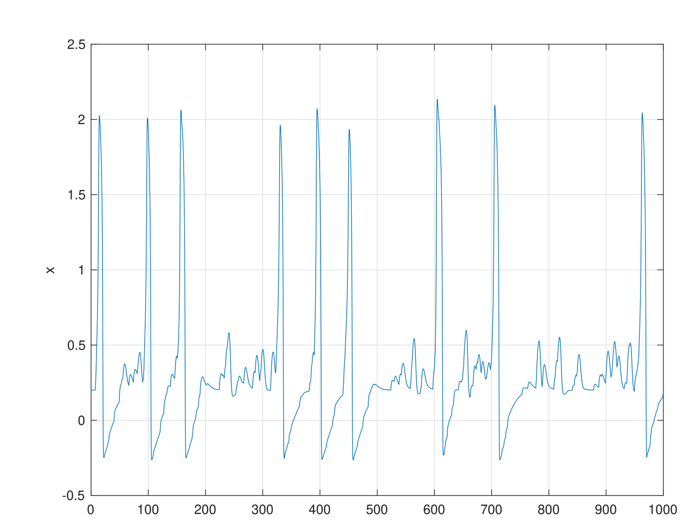
When $\bar{\tau} = 1$, kicks happen so frequently that excursions happen with near certainty. See Figure 5(b).



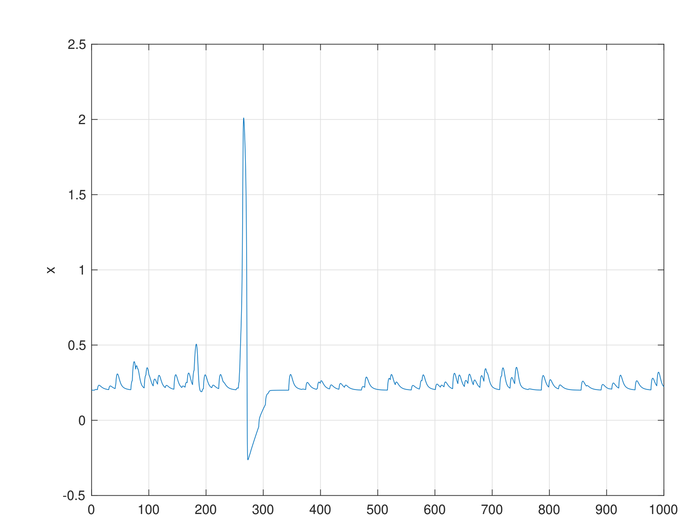
(a) Probability of excursion vs. $\bar{\tau}$



(b) $\bar{\tau} = 1$



(c) $\bar{\tau} = 4$



(d) $\bar{\tau} = 10$

Figure 5. Pane (a) shows the dependence of the probability of excursion on $\bar{\tau}$. Panes (b), (c), (d) show typical time series plots for x with various values of $\bar{\tau}$.

The time until an excursion happens, commonly referred to in the literature as the **interspike interval**, is believed to be exponentially distributed. In Figure 6 (left), we let $\bar{\tau} = 5$ and use maximum likelihood estimation, implemented with `histfit` in MATLAB, to fit interspike interval data to an exponential density (red curve) with $\bar{\sigma} = 111.38$ with 95% confidence interval $\bar{\sigma} \pm 2.1835$. Figure 6 (right) also gives a graphical justification for fitting our data to the exponential distribution by using a Q-Q plot. The linearity of the data points suggests that our data falls into the theoretical quantiles of an exponential distribution. Similar fits hold with other values of $\bar{\tau}$ in our experiments.

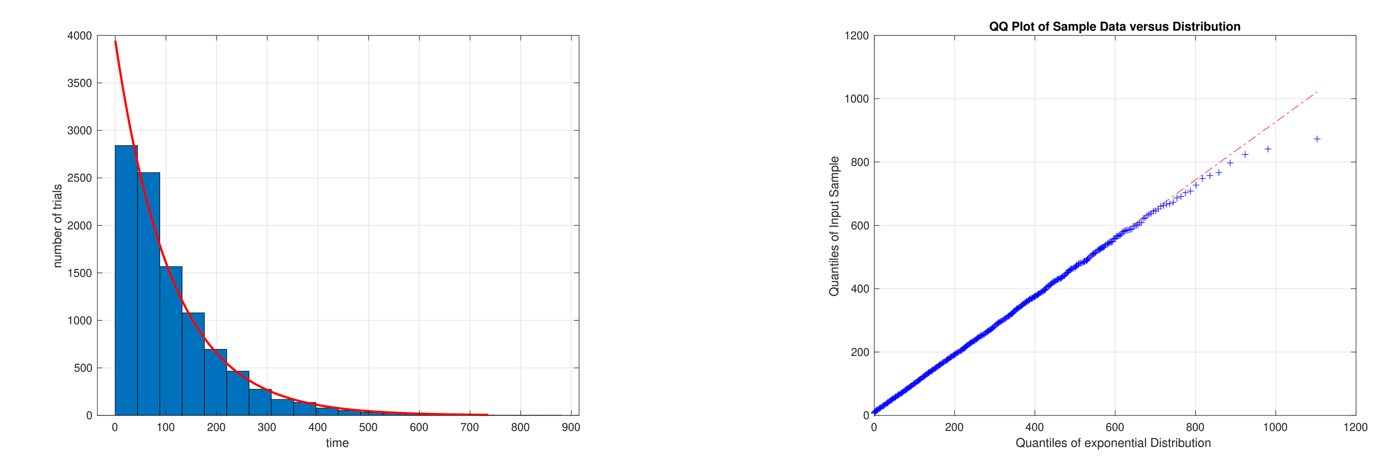


Figure 6. Interspike interval σ data

When an excursion occurs, there can be multiple trips around the limit cycle (ie. consecutive spikes occur as in Figure 5c) or only one (ie. spikes are isolated as in Figure 5d). We call the time χ spent orbiting a limit cycle the **excursion time**. The distribution of χ is qualitatively different for various values of $\bar{\tau}$. When $\bar{\tau} = 4$, the probability distribution of χ is concentrated around multiple values, indicating multiple consecutive spikes. When $\bar{\tau} = 10$, the probability distribution is concentrated around a single value, indicating isolated spikes. See Figure 7.

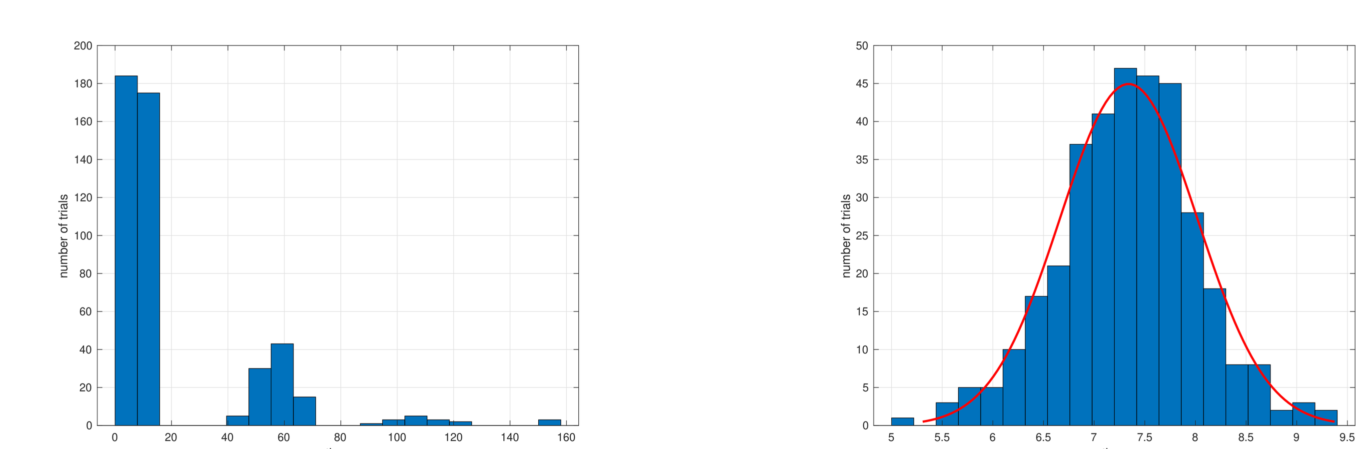


Figure 7. Histogram of χ for $\bar{\tau} = 4$ and $\bar{\tau} = 10$

Future directions

- Develop mathematical frameworks that can explain the observed behavior of σ and χ under varying conditions on τ.
- Analyze further statistical properties of the system, such as the number of excursions in each limit cycle.

References

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